Introduction to General Relativity

With the “Twin Paradox” Story

Special Relativity: Perfect Inertial Frame

No acceleration.

No gravitational field.

Constant velocity in straight line.

(Free floating in intergalactic space).

Special Relativity: Lab Frame

APPROXIMATELY

No acceleration.

No gravity.

Constant velocity in a straight line.

(A typical lab on Earth).

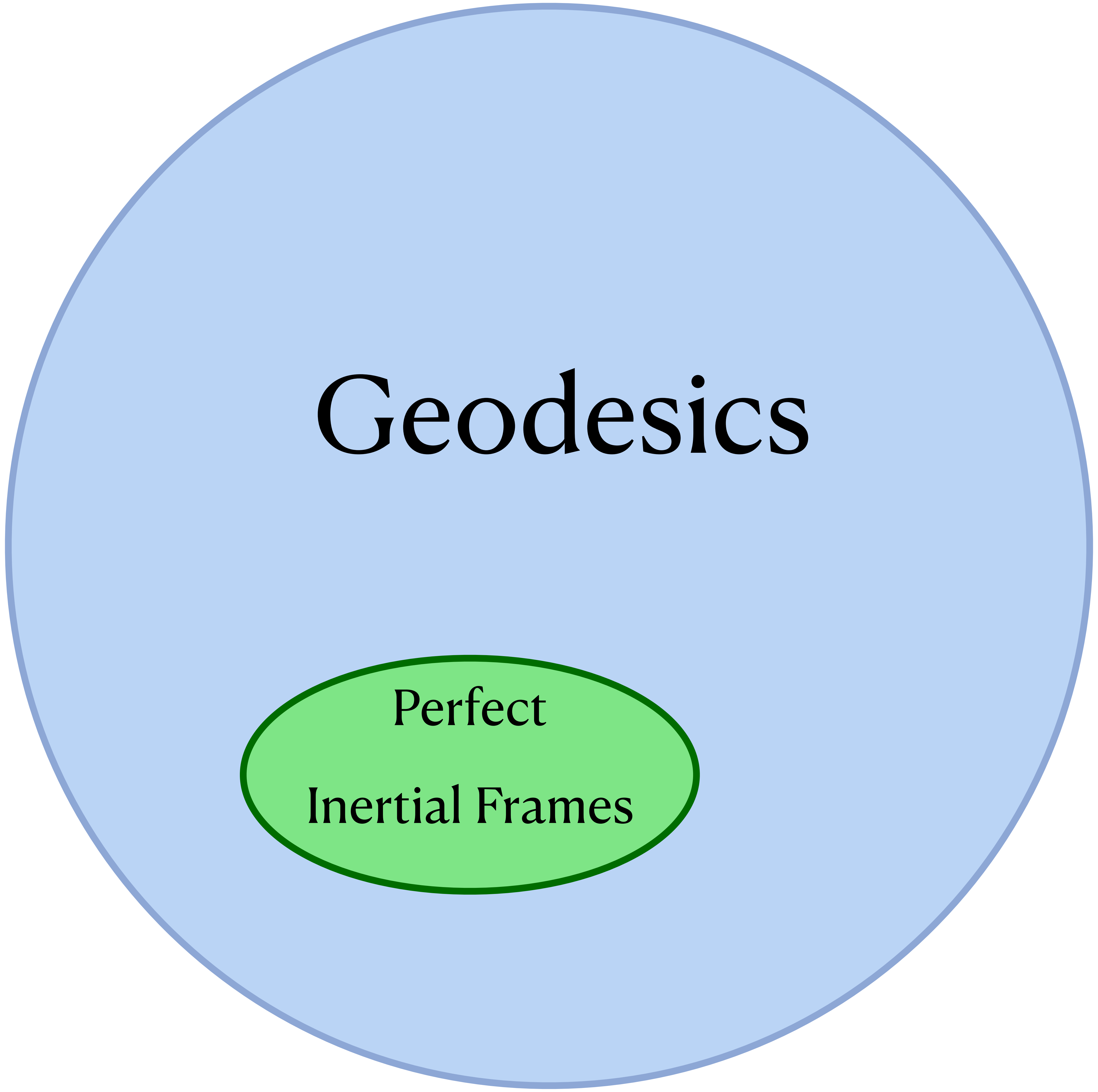
General Relativity: Geodesic

In General Relativity, a geodesic is the straightest possible path through curved space.

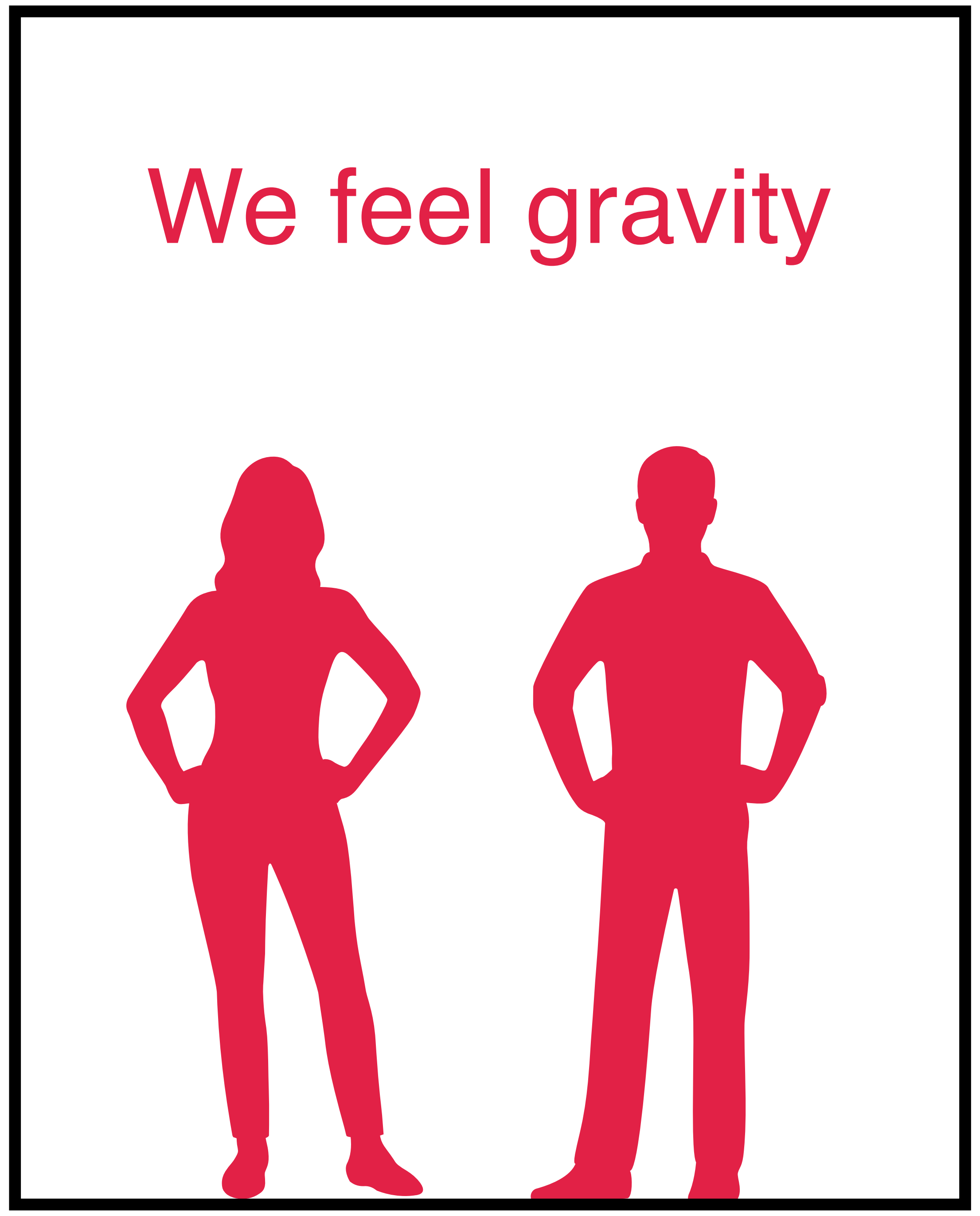
Motion on a geodesic includes perfect inertial frames,

PLUS

Free fall in a gravitational field.


Geodesics
Perfect
Inertial Frames

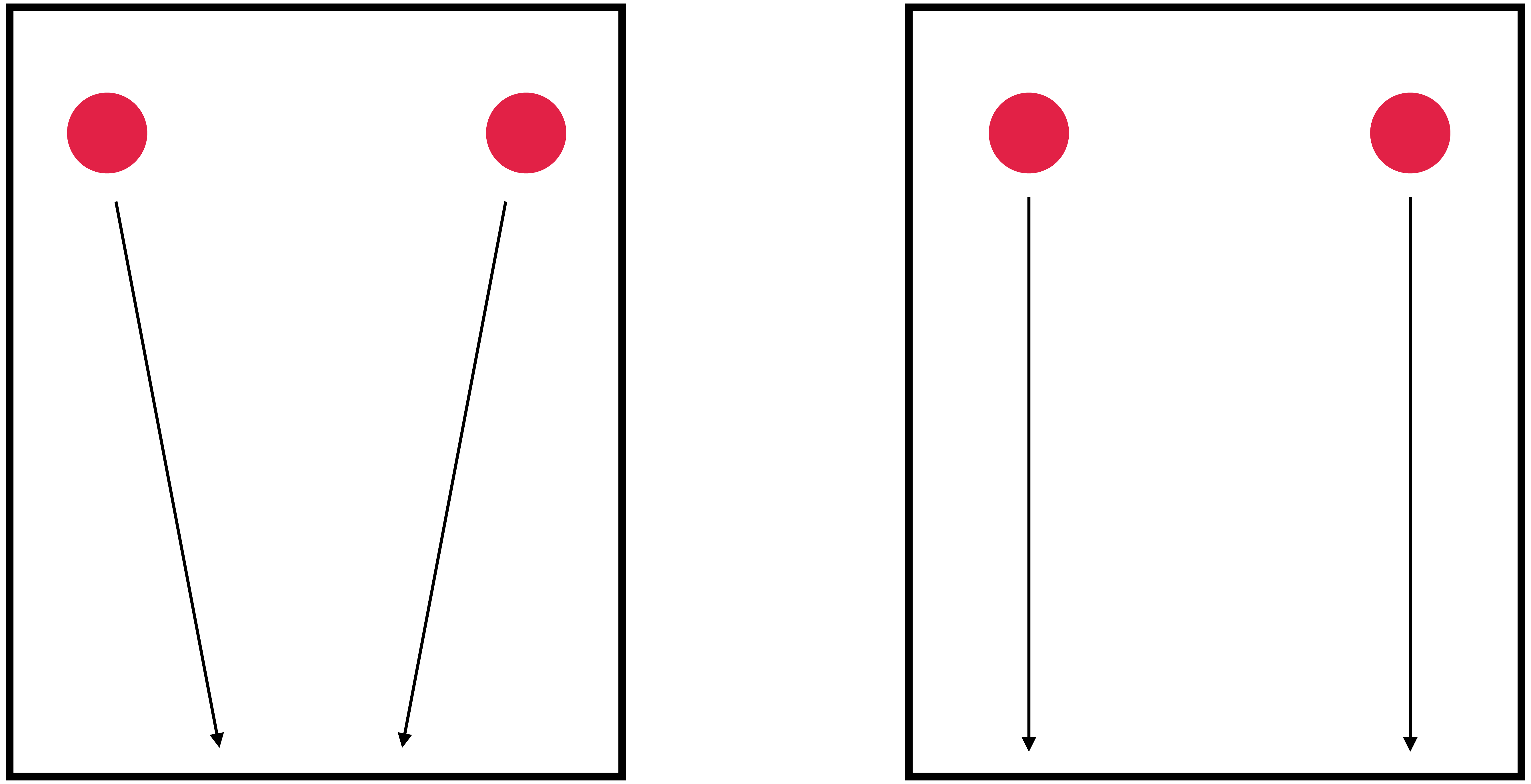
Equivalence Principle with Elevators



At rest on Earth or accelerating through space?

Floating through space or in free fall toward Earth?

Tidal Forces, BIG Elevators

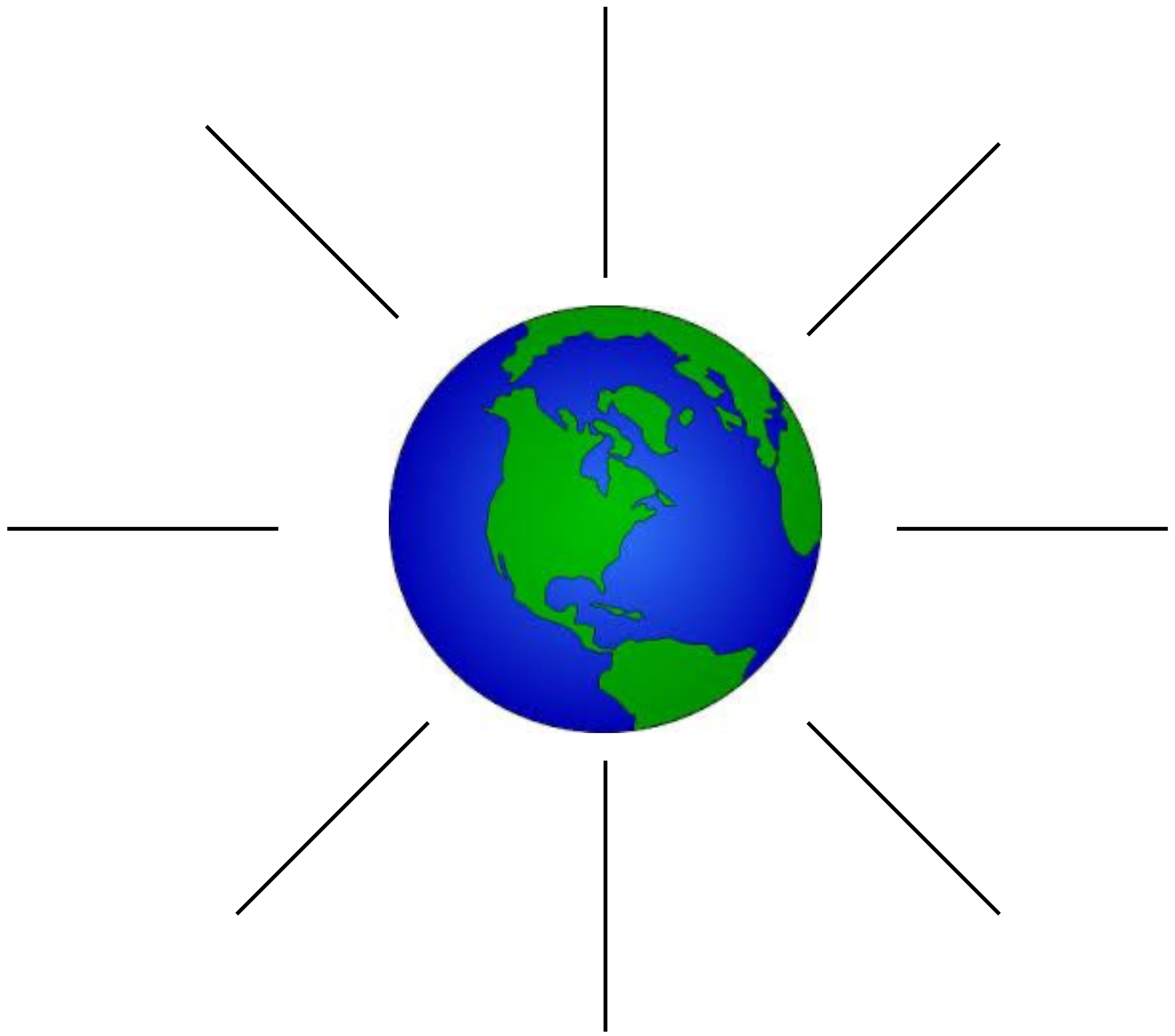


At rest on Earth

Accelerating
through space

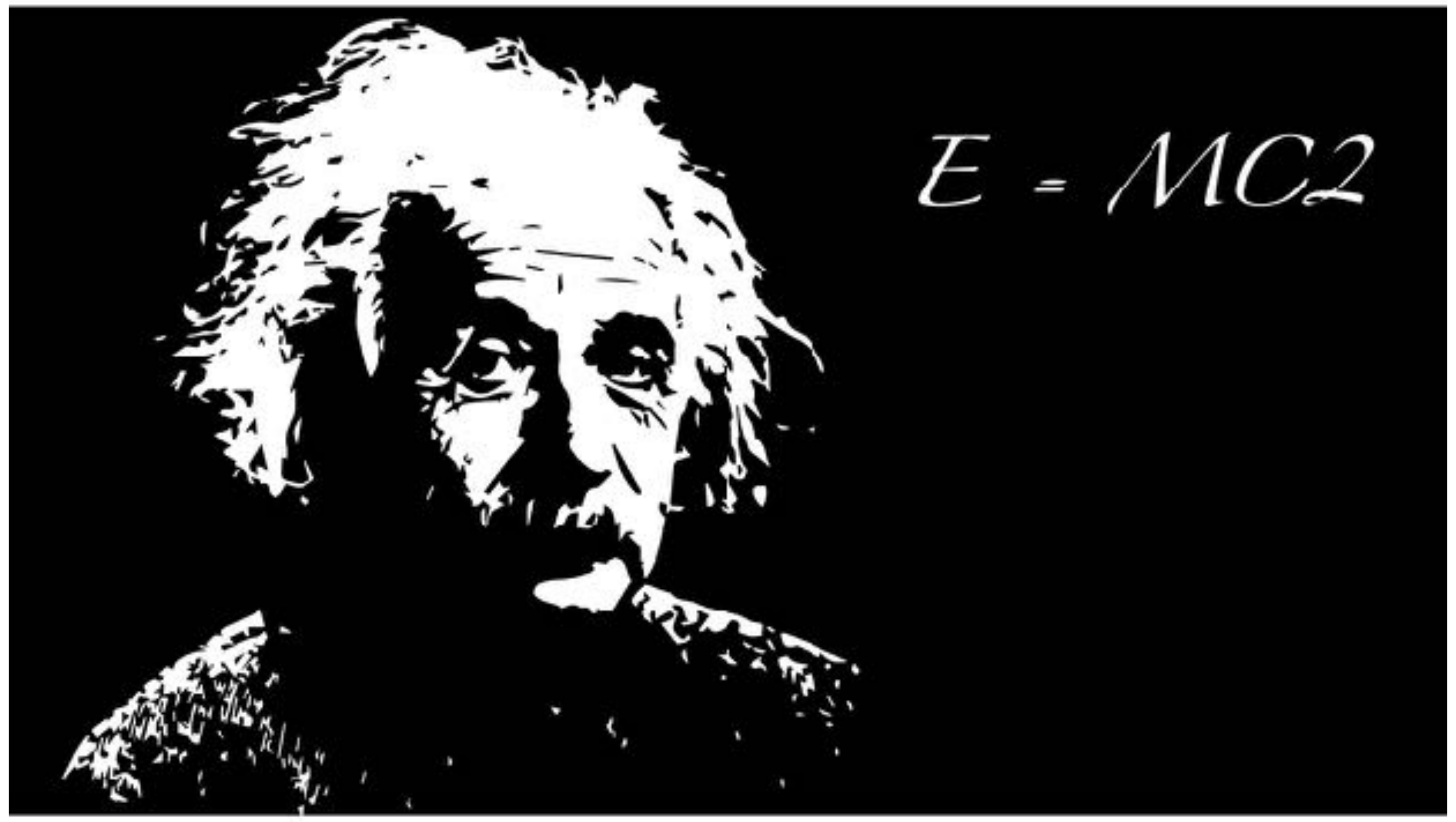
Acceleration vs. Gravity

- Acceleration has no tidal forces.
- It has the same effects as a PERFECTLY UNIFORM gravitational field, which does not exist in reality.



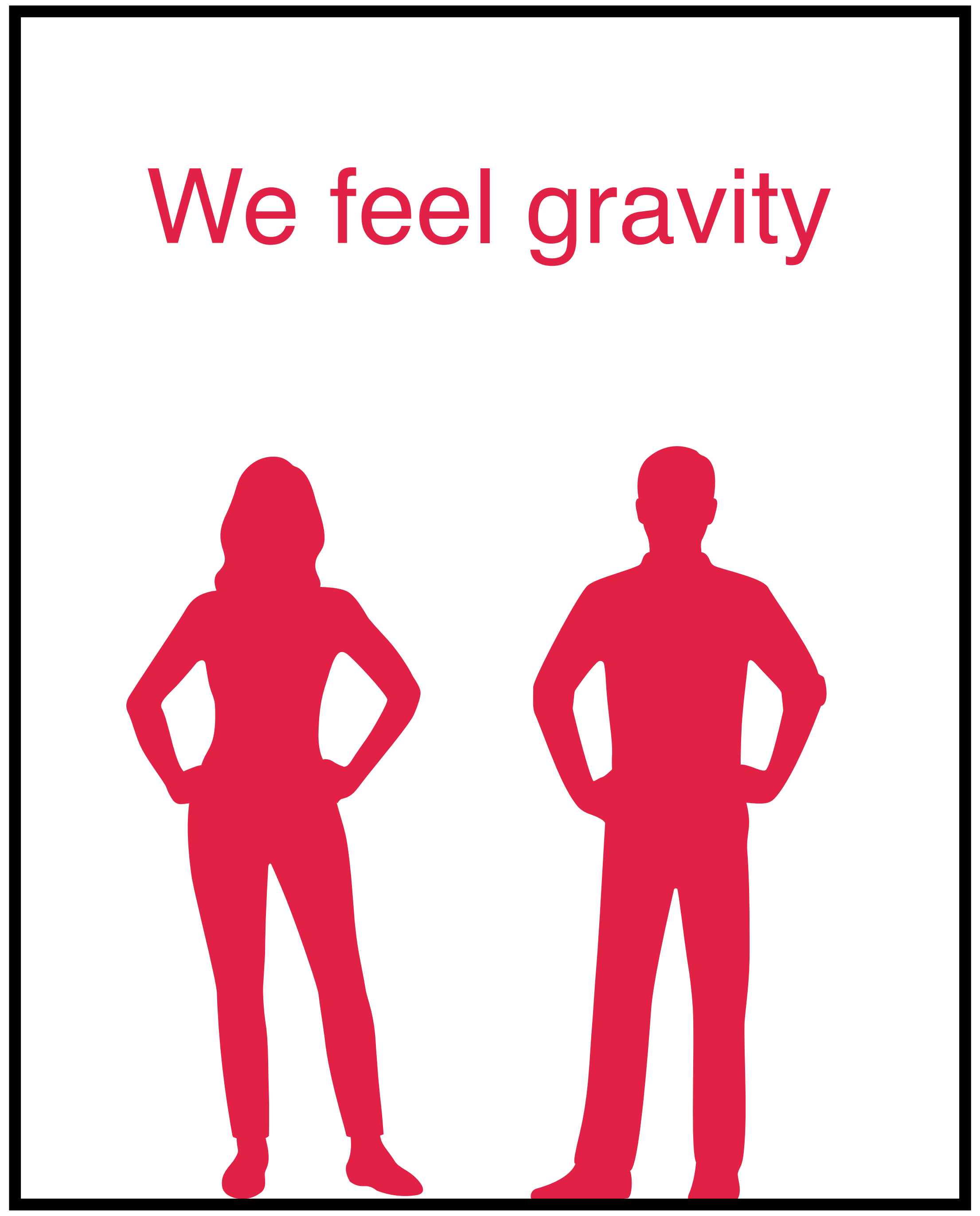
Equivalence Principle

This is what gave Einstein the idea that space is curved by gravity.



Most Confusing Part of General Relativity

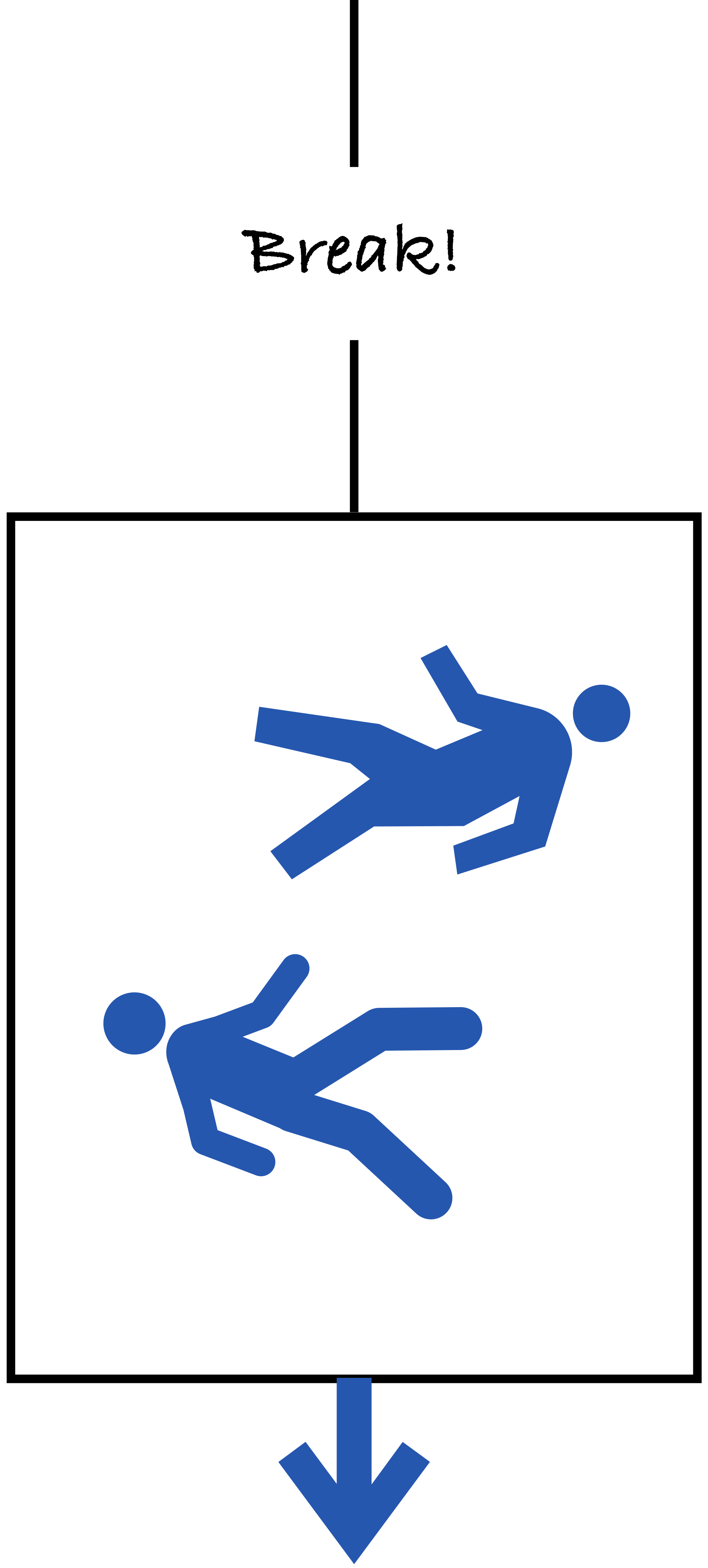
This one is
the lab frame



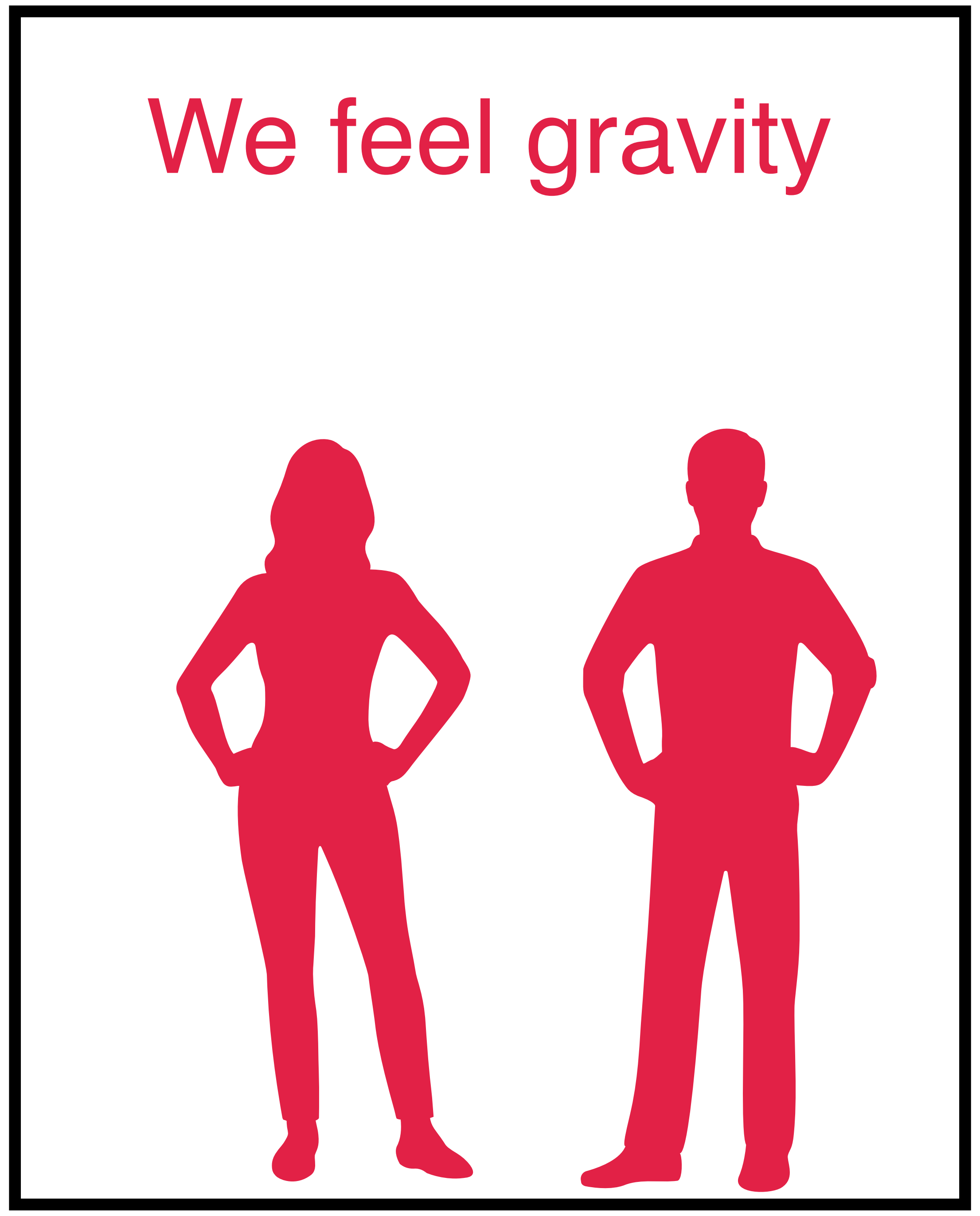
At rest on Earth

This one is
the geodesic

In free fall
toward Earth



“Obviously” accelerating



"Obviously" with gravity,

So not perfect inertial frame.

Earth Labs

For **relativistic** calculations,

Earth's rotation ≈ 0

Earth's velocity in orbiting the sun ≈ 0

Sun's velocity in orbiting the galaxy ≈ 0

Earth's gravity ≈ 0

Earth Labs

We are used to neglecting motion of Earth's surface.

We have to get used to neglecting Earth's gravity in MOST relativistic calculations

(A notable exception is the GPS satellite clocks).

Earth Labs

For relativistic calculations,

All the components of Earth's velocity $\lll c$, relative to a point at the center of the galaxy

(or anything we are likely to compare it to).

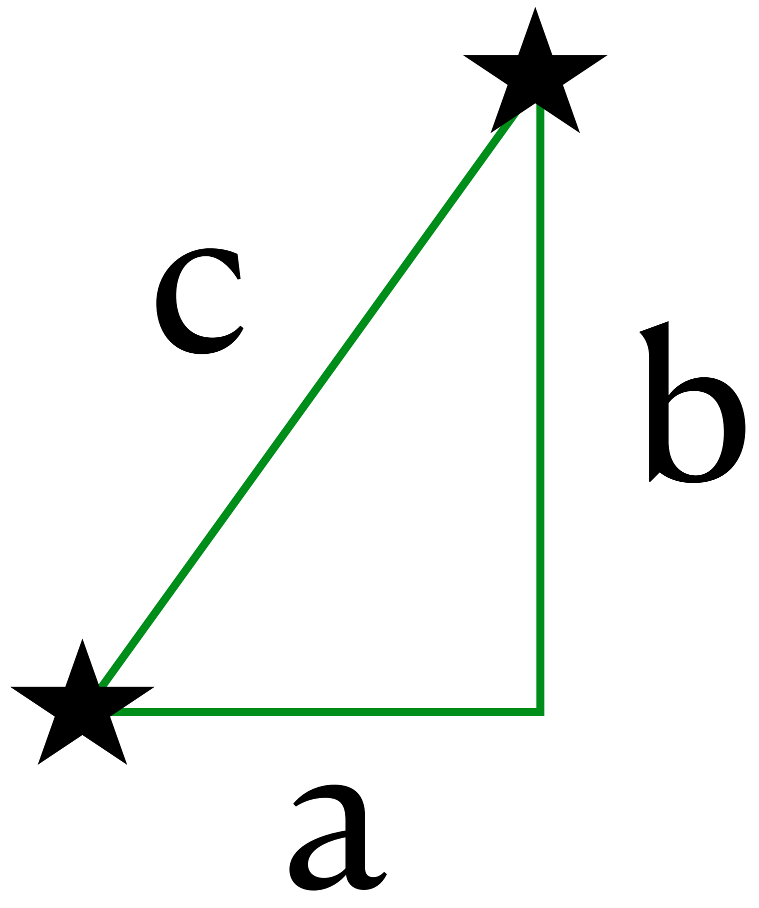
Earth's gravity slows clocks by

$$\Delta\tau_E = 0.9999999993\ \Delta t$$

Relative to perfect inertial frame

General Relativity

It's all about the metric, which tells us how to calculate "distances" between two events.



$$c^2 = a^2 + b^2$$

General Relativity

It’s all about the metric, which tells us how to calculate “distances” between two events.

$$dS^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}\phi^2 + g_{03}dtd\phi$$

Metric elements

Spacetime Interval

(Squared)

The Kerr metric has a

cross term for rotating

Black Holes

Spacetime Intervals

- Introductory physics courses may not tell you much about spacetime intervals, because you can work a lot of problems using the Lorentz transformations.
- But when we get to General Relativity, spacetime intervals are absolutely necessary. That's how we "navigate" curved spacetime.

General Relativity

Einstein's field equations are a set of differential equations for the metric elements.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

Metric elements

There are two commonly used solutions.

Friedman-Lemaitre-Robertson-Walker Metric
(uniform isotropic space between galaxies)

$$dS^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Scale factor for expansion of the universe

Overall curvature of the universe

Schwarzschild Metric (effects of gravity)

Schwarzschild radius

$$dS^2 = -(1-\frac{R_S}{r})c^2dt^2 + \frac{1}{1-\frac{R_S}{r}}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

Friedman-Lemaitre-Robertson-Walker metric, for $a = 1$ and $k = 0$

$$dS^2 = -c^2dt^2 + \cancel{a}^{\,1}\,{}^2(t)\left[\frac{dr^2}{1 - \cancel{k}_{0}\,r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Schwarzschild metric, for $r \ggg R_S$

$$dS^2 = -\left(1 - \cancelto{0}{\frac{R_S}{r}}\right)c^2dt^2 + \frac{1}{1 - \cancelto{0}{\frac{R_S}{r}}}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

Minkowski Metric (spherical coordinates)

$$dS^2 = -\, c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Minkowski Metric, Cartesian coordinates

$$dS^2 = -\,c^2dt^2 + (dx^2 + dy^2 + dz^2)$$

How are we using General Relativity?

- General Relativity tells us to integrate the proper time from the spacetime interval to get the time we experience.
- Also it shows us how to calculate the effects of Earth gravity (as in the GPS video):

$$\Delta\tau_E^2 = \left(1 - \frac{R_s}{r_E}\right)\Delta t^2$$

$$\Delta\tau_E = 0.9999999993\ \Delta t$$

Proper time (measured by clock you carry with you)

$$c^2 d\tau^2 = -\,dS^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Twin Paradox Math

Time you experience

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - (dx^2 + dy^2 + dz^2)}$$

Time in lab frame

Change in your position
as measured in lab frame

Time you experience (infinitesimal)

Time in lab frame

Minus A positive number

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2 - \overbrace{(dx^2 + dy^2 + dz^2)}}$$

Change in your position

As measured in lab frame

Integrate to get total time you experience

$$d\tau = \frac{1}{c}\sqrt{c^2 dt^2 - (dx^2 + dy^2 + dz^2)}$$

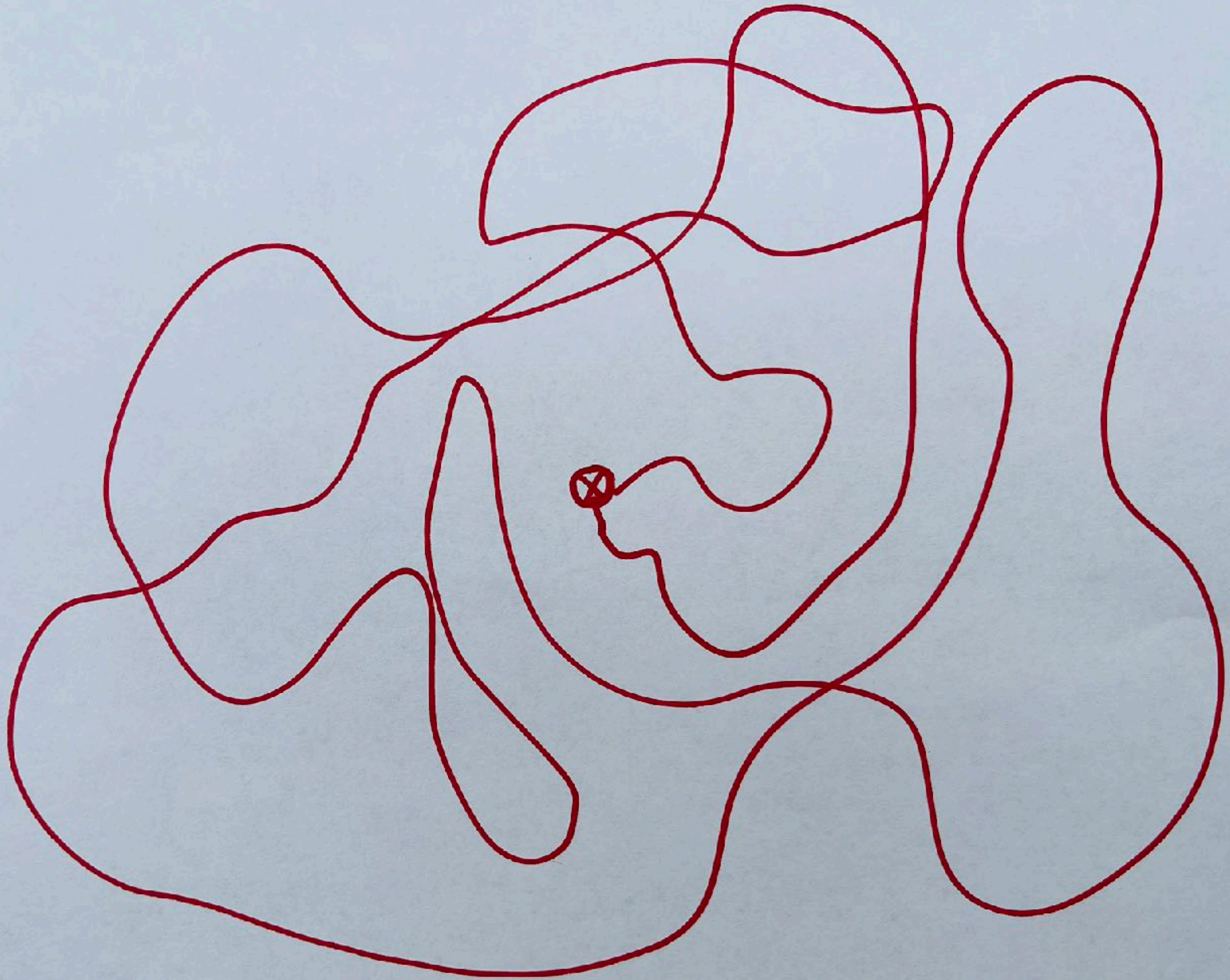
$$\int d\tau = \int dt \sqrt{1 - \frac{1}{c^2}\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)}$$

$$\int d\tau = \int dt \sqrt{1 - \frac{1}{c^2}\left(v_x(t)^2 + v_y(t)^2 + v_z(t)^2\right)}$$

$$\int d\tau = \int dt \sqrt{1 - \frac{1}{c^2}\left(\overrightarrow{v(t)}^2\right)}$$

IF v is constant,
we get the familiar
Time dilation from
Lorentz equations:

$$\tau = t\sqrt{1 - \frac{v^2}{c^2}}$$



For $v(t)$ not constant, we just get a more complicated integral

$$\int d\tau = \int dt \sqrt{1 - \frac{1}{c^2}\left(\overrightarrow{v(t)}^2\right)}$$

We can always evaluate it numerically.

Longest Time

In Special Relativity, clocks show the longest time between two events when they remain in the Lab Frame.

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2 - (\cancelto{0}{dx^2} + \cancelto{0}{dy^2} + \cancelto{0}{dz^2})}$$

$$d\tau = \frac{1}{c}\sqrt{c^2dt^2} = dt$$

Longest Time

In General Relativity, clocks show the longest time between two events when they travel on a Geodesic.

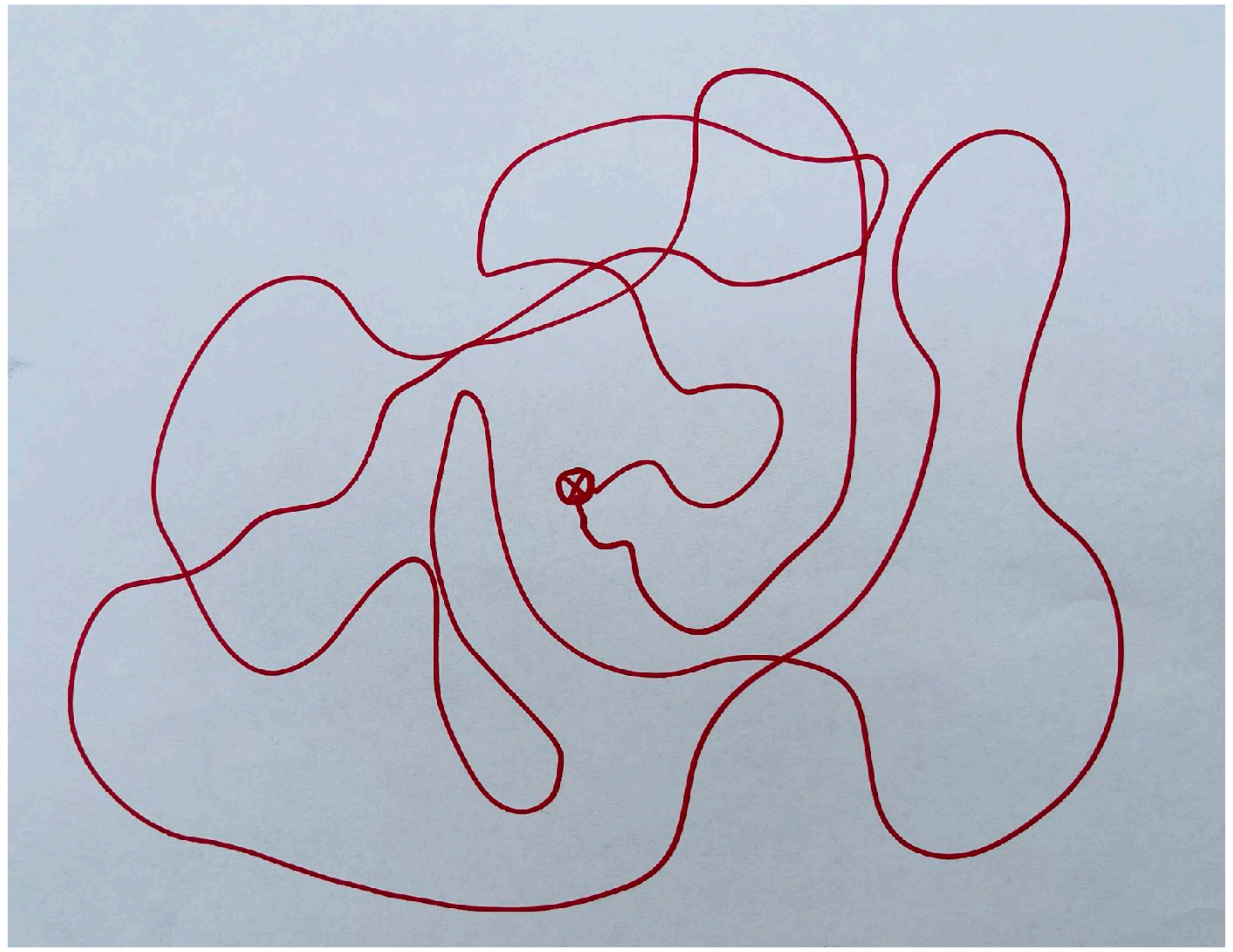
This is more general than the perfect inertial frame, for which the lab frame is a good approximation.

Back to the Twins

- If twins go on separate journeys and then meet again to compare clocks, the one who spent more time on a geodesic (such as at home) will have more time elapsed.

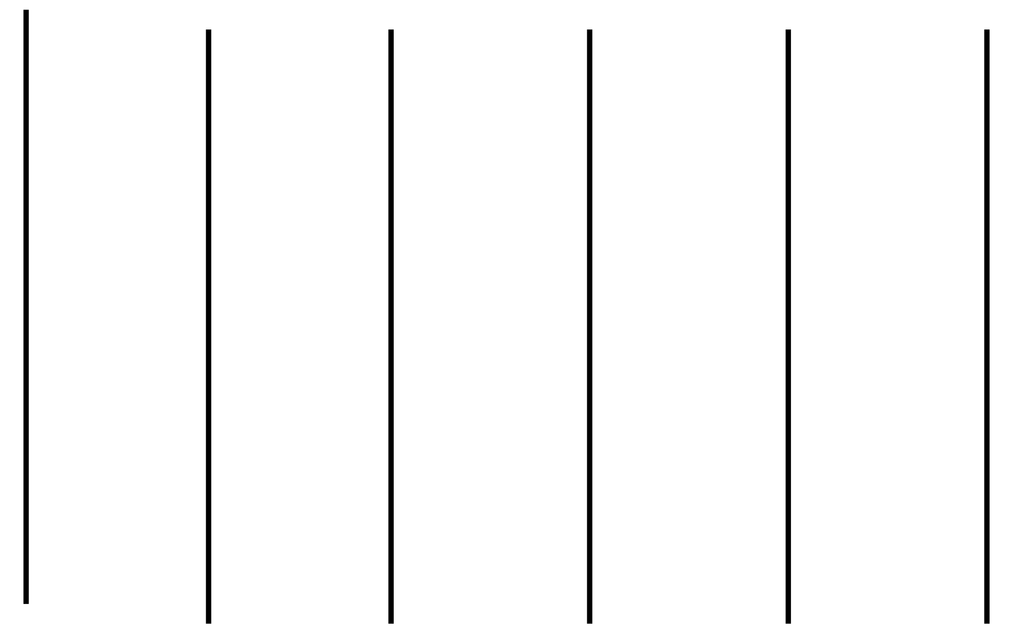
Acceleration?

- Notice the changes in velocity indicate acceleration, but the acceleration doesn't do anything beyond changing the velocities.

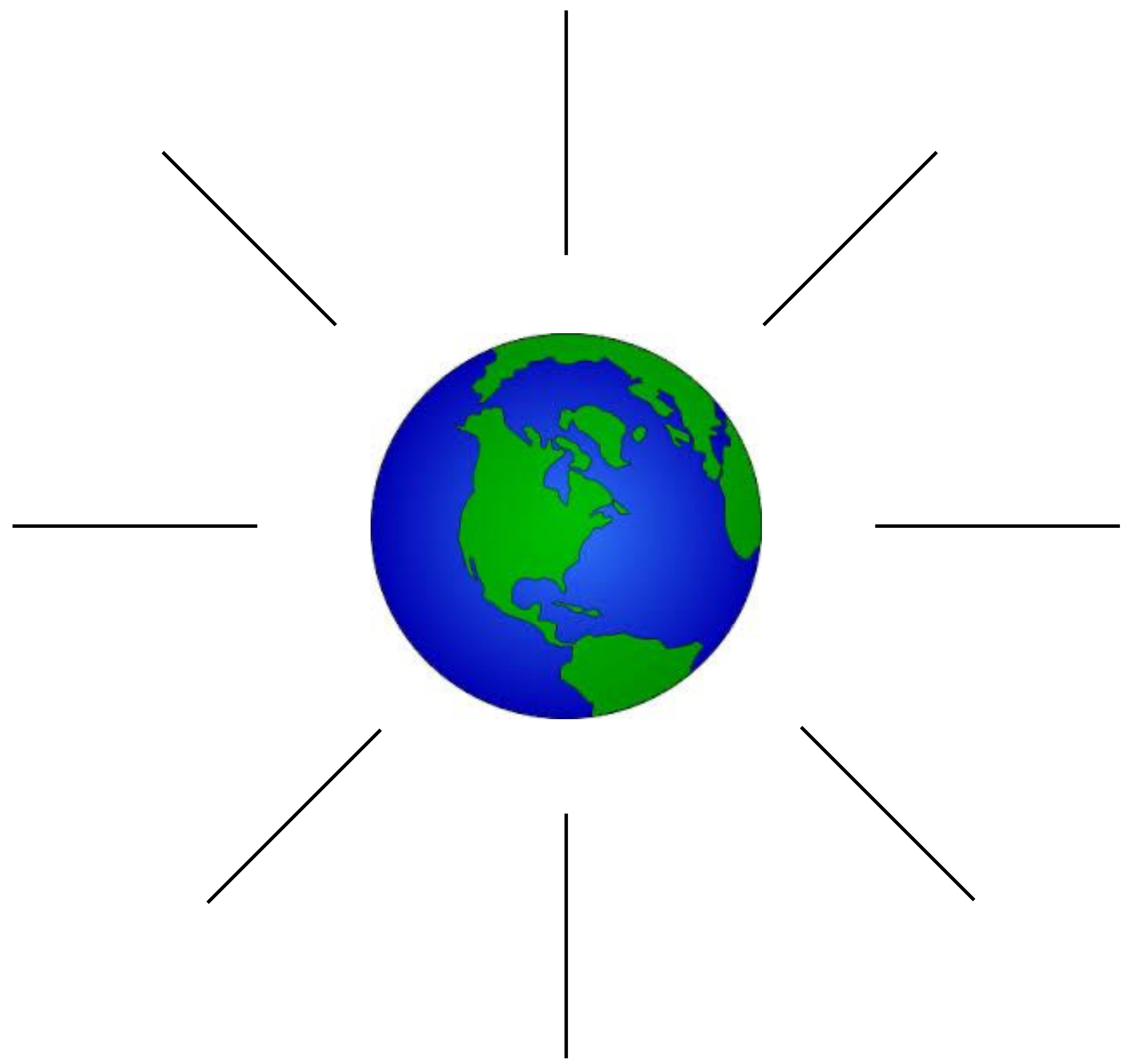


Why No Term for Acceleration?

- Uniform Gravitational Field



- Real Gravitational Field



Acceleration?

- “Does acceleration explain the twin paradox?”
- “Does acceleration **cause** one twin to age faster than the other?”
- “Is the twin paradox about acceleration?”
- These are all trick questions that have kept people arguing about relativity for at least 62 years. The arguments seem to keep a lot of YouTube creators in business.

My Answer?

- Go out and travel!
- No matter how tiny the relativistic effect is, it does keep you younger!