Lecture Notes on Einstein's Special Relativity: What You See Is Not Always What You Get

Linda Fugate

March 18, 2024

1 Introduction

Einstein's theory of relativity is a fun upgrade to classical physics. Special relativity covers electromagnetism, and general relativity covers gravity. The speed of light is a key part of electromagnetism, so special relativity is the place where we can play around (mathematically) with rockets traveling at close to the speed of light.

Relativity includes a lot of surprises. Space and time are no longer what we expect them to be. Hermann Minkowski, who was Albert Einstein's professor and later collaborator, commented [1]:

> "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Our brains are not wired to perceive the union of space and time, so the results of relativity are not what we expect. Here's another quote from Tim Maudlin, Professor of Philosophy [2]:

> "Special relativity is a very simple theory that is commonly presented in a complex and confusing way"

I agree completely. This paper and the corresponding seminar introduce some of the surprises of special relativity. I hope my presentation is less complex and confusing than usual.

2 Light Travel Time

Suppose Alice and Bob set up a relativity experiment. Bob is going to fly a rocket at 60% of the speed of light, while Alice observes with a telescope on Earth. They set up signposts spaced one light hour apart. One light hour is the distance light travels in an hour, and

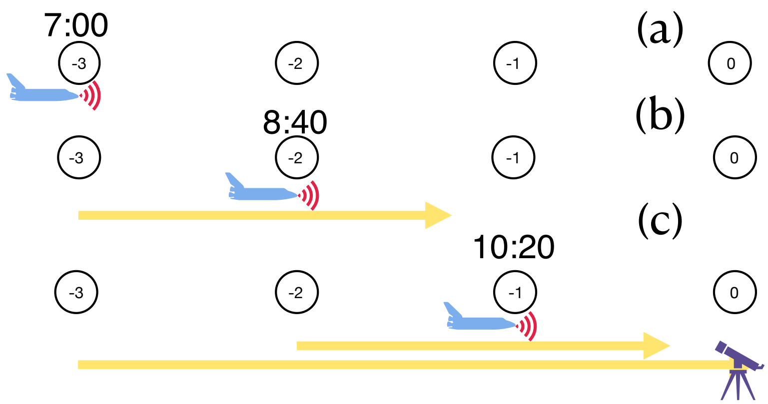

Figure 1: Progress of Bob's Rocket and the Light Signals at Three Times: (a) 7:00 am, (b) 8:40 am, and (c) 10:20 am

is almost the distance from Earth to Saturn. I don't know how they could set up their signposts, but let's just assume they did. They plan for Bob to accelerate up to 0.6c a long way away, then fly at constant speed past the signposts. As he passes each signpost, he will send a light signal to Alice. They use seven signposts, one directly over Alice's observatory, and three on each side. They arrange the experiment so that Bob will pass the first signpost and send the first signal at 7:00 am according to Alice's time.

Figure 1 shows the progress of Bob's rocket and the light signals at three different times. He's traveling at 0.6c = (3/5)c with respect to the signposts, so it takes him 5/3 hour = 1 hour 40 minutes to travel 1 light hour. The light signals travel at the speed of light, obviously, so it takes 3 hours for the first signal to reach Alice's telescope, 2 hours for the second one, and 1 hour for the third one. His first light signal reaches Alice's telescope before he reaches the third signpost, but his second signal is still in flight.

Table 1: Arrival Times of First Three Light Signals

Signal	Time Transmitted	Time Received
1	7:00	10:00
2	8:40	10:40
3	10:20	11:20

Table 1 shows the arrival times of the first three light signals at Alice's observatory.

Our everyday experience leads us to believe we see things when they happen, so it appears that Bob travels one light hour in only 40 minutes. This would make his velocity faster than the speed of light, (5/3)c to be exact. Of course that is not what happens. Alice has to correct for the light travel time to see that Bob is actually traveling at (3/5)c, as they planned the experiment.

Some astronomical observations turn up jets from quasars that appear to be moving faster than light in a direction perpendicular to our line of sight. Astronomers interpret this to mean the jet is oriented diagonal to our line of sight, and moving close to the speed of light. The correction for light travel time clears up the apparent faster-than-light problem.

Figure 2 shows the progress of Bob's rocket after he passes Alice's observatory at 12:00 noon. He passes the last three signposts and sends his final three signals at 1:40 pm, 3:20 pm, and 5 pm. Now each light signal reaches Alice's telescope before Bob reaches the next signpost.

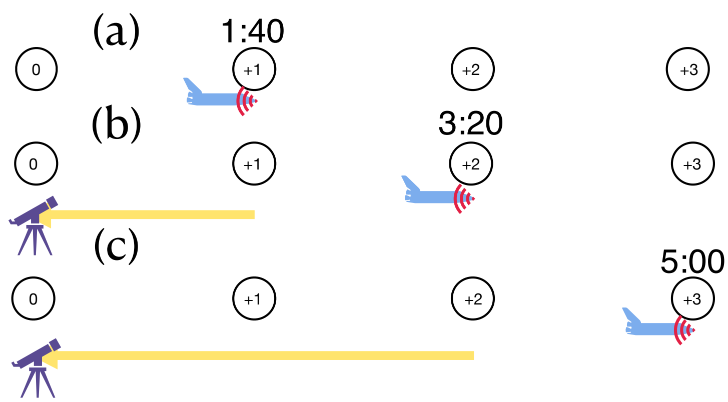

Figure 2: Progress of Bob's Rocket and the Light Signals at Three Times: (a) 1:40 pm, (b) 3:20 pm, and (c) 5:00 pm

Table 2 shows the arrival times of Bob's last three signals. Now, if we do not correct for light travel time, it looks like Bob has slowed down. The arrival time between signals sent 1 light hour apart is 2 hours, 40 minutes = (8/3) hour. It appears that Bob is going only (3/8)c, instead of (3/5)c. However, correction for light travel time shows that nothing has changed; Bob is traveling at a constant 60% of the speed of light as planned.

3 The Heart of Special Relativity

From Figure 1, it appears the light signal is outrunning the rocket by not very much. So would Bob see the signal moving ahead of him at only 40% of the speed of light (1.2×10^8

Table 2: Arrival Times of Last Three Light Signals

Signal	Time Transmitted	Time Received
1	1:40	2:40
2	3:20	5:20
3	5:00	8:00

m/s, since he's going 60% of that speed (1.8×10^8 m/s)? Well, no; first of all, we can't see light moving away from us; we see it only when it comes into our eyes, cameras, photodetectors, etc. But suppose we add another rocket, with Charlie going the same speed as Bob, with a head start of one signpost. See Figure 3. Bob can send a signal to Charlie, and ask him how long it takes him to receive it. Now Alice sees the signal catching up with Charlie, but it approaches Charlie only 40% as fast as it approaches the signposts which are not flying away from it.

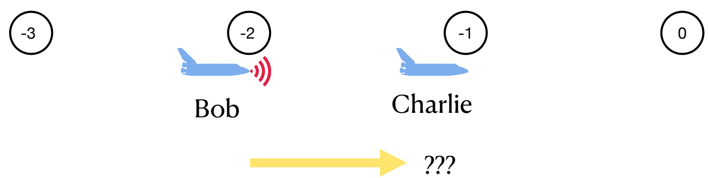

Figure 3: Bob and Charlie are both traveling at 60%c.

For sound waves, the speed of sound is measured relative to the medium it is traveling through, such as air or water. Light travels through vacuum. If vacuum worked the same way for light as air does for sound, then the light ray would be moving at only 40% of the speed of light relative to both Bob and Charlie, who are one light hour apart. Thus it would take 2.5 hours for Charlie to receive the signal. Here I'm using the common sense (but incorrect) assumption that the vacuum is a fixed reference frame at rest with respect to us here on Earth.

However, Einstein told us the speed of light is the same in all inertial reference frames.

For a simple definition of ”inertial reference frame”, it’s a set of coordinates where Newton’s laws and our common sense are correct for low velocities. So when Bob calculates the speed of the light signal he sent Charlie, he will get the same value of c that Alice sees. This should be surprising to everyone who has ever thought about velocities. The only way to make it work is to expand our understanding of geometry.

4 The Light Clock

There are several ways to derive the equations of special relativity [3, 4, 5]. Here I’ll use the light clock, which is a hypothetical device with a set of mirrors to bounce a light ray up and down, measuring the time between bounces from the bottom mirror. So let’s say Alice and Bob each have a light clock, with the mirrors 1.5 m apart. That makes the round trip 3 m. The speed of light is 3×10^8m/s, so the clock ticks once every 10^{-8}s. Each clock comes with a counter which registers the total number of seconds since it was last reset.

First Alice and Bob compare their light clocks side by side, to make sure they work the same way. Then Bob takes his clock to his rocket garage, and prepares for another run past the signposts. During this second run, Bob observes his clock working the same as it did back in the lab with Alice; the light ray just goes up and down between the two mirrors.

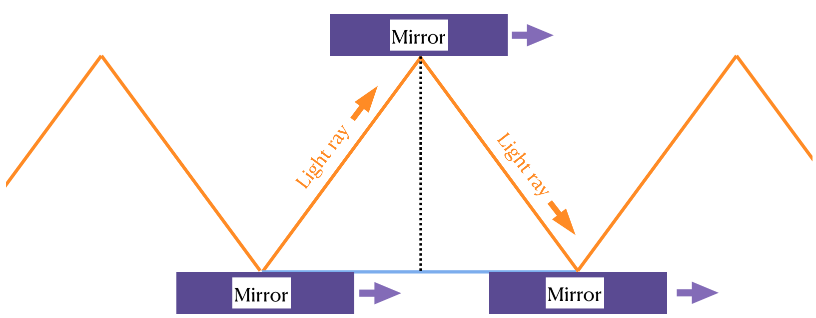

Figure 4: Alice’s View of Bob’s Light Clock

This time Alice watches Bob’s light clock with a super duper telescope, so she can see the counter on Bob’s clock. She notices that the path of the light ray has to be a zigzag, as shown in Figure 4, since the clock is moving with the rocket. During one tick of Bob’s clock, $\Delta\tau = 10^{-8}$ seconds, the mirrors move some distance to the right, so the light ray travels farther than the 3 m it travels (just up and down) in Bob's frame. Therefore, the time between ticks must be longer in Alice's frame. This is equivalent to saying Alice sees Bob's clock run slower, and it's called time dilation. How much slower? We can calculate from the triangle in Figure 5, which shows half of one clock tick:

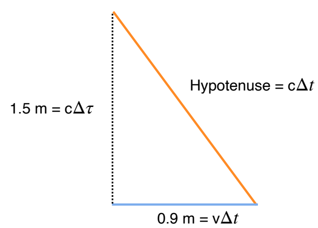

Figure 5: Calculating Light Path

$$(c\Delta t)^2 = (v\Delta t)^2 + (c\Delta\tau)^2 \tag{1}$$

$$\Delta\tau^2 = (c^2 - v^2)\Delta t^2/c^2 = (1 - v^2/c^2)\Delta t^2 \tag{2}$$

$$\Delta\tau = \sqrt{(1 - v^2/c^2)}\Delta t = \Delta t/\gamma \tag{3}$$

Or

$$\Delta t = \gamma\Delta\tau \tag{4}$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. This is the first of the Lorentz transformations. For $v/c = 0.6$,

$$\gamma = \frac{1}{\sqrt{1 - 0.36}} = \frac{1}{\sqrt{0.64}} = \frac{1}{0.8} = 1.25 \tag{5}$$

And $1/\gamma = 0.80$. Alice sees Bob's clocks running 80% as fast as hers.

Now what about the light signal Bob sends to Charlie? It also has to travel at the speed of light. So the distance Bob measures between signposts must be shorter by the same ratio that his clock is slower, compared to what Alice measures. This transformation is called length contraction. For a length L in Alice's frame,

$$\Delta L(\text{frame with time } \tau) = \Delta L(\text{frame with time } t)/\gamma \tag{6}$$

In my opinion, this would be easier to remember if γ had been defined so that we *multiply* by γ instead of *divide* by γ in Equations (3) and (6), because we usually think of the frame with time t as the one we're in, while the frame with time τ is the one with a rocket flying by at velocity close to the speed of light. However, the definition of γ is well established, so we just have to remember it.

5 What You See Is Not Always What You Get

As a shortcut, we often say things like "moving clocks run slower" and "moving objects have length contraction". Is this real? No. The point of relativity is that all inertial frames are equivalent. Bob and Charlie have their own rest frame, in which Alice and her signposts are moving at 60% of the speed of light in the opposite direction. And Bob will see Alice's clock slow down, exactly the same way Alice sees Bob's clock slow down. This sounds impossible, and it's the basis for the well-known twin paradox story that we will look at in the next lecture. To see how it is possible, we need to think about how we decide one clock is running slower than another one. The obvious way is to put them together and set them to the same time, then check both of them later *at the same time*.

For our story of Bob and Alice, there is only one event when they are together, at 12:00 noon when Bob flies directly over Alice's observatory. That's when they can synchronize their clocks. But as long as both stay in their own inertial frames, they will never again be together. If Bob turns around and returns home (as we plan for him to do), he has to leave his inertial frame, and the equations we looked at in this paper are not sufficient to calculate what happens to his clock. As it turns out in relativity, when two reference frames are moving at constant velocity with respect to each other, there can never be more than one event that is simultaneous in both frames.

This is illustrated in Minkowski diagrams, as shown in Figure 6. The ordinary black perpendicular axes represent time and space for the rest frame of an observer. The units are chosen such that the speed of light is one (for example, one lightyear per year), and the path of a light ray is a 45° line. The colored axes represent the time and space axes for frames moving at velocities of 0.2c, 0.5c, 0.8c, and 0.9c relative to the rest frame. Events along the space axis, or any axis parallel to the space axis, are simultaneous in the given frame. So only one event can be simultaneous in two different frames, because there can be only one intersection of two lines with different slopes. We don't notice this effect in ordinary life, because we encounter only velocities much less than the speed of light.

Minkowski diagrams take a little getting used to. We will look at them in more detail for the next talk, about the twin paradox. They tell us relativity is somewhat like rotating the axes of ordinary Euclidean geometry, but the time and space axes rotate toward each other to keep the speed of light constant.

The important thing to remember about time dilation and length contraction in special relativity is that these are just coordinate issues. Our brains are not wired to perceive mixing of time and space coordinates, so the results are very surprising. However, we can keep in mind the fact that if Bob flies over us in a rocket right now, we are not going to age more slowly or get thinner. These coordinate transformations are just how we measure things in a different reference frame. However, when we change reference frames, things get real and we *can* age more slowly.

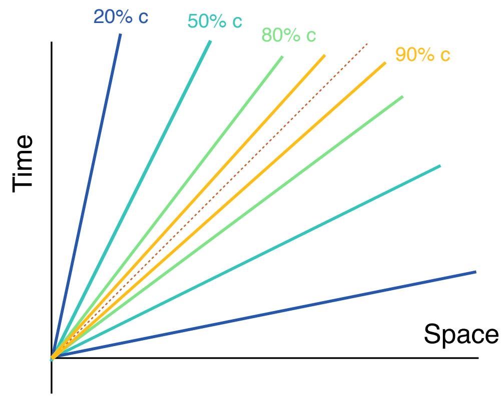

Figure 6: Minkowski Diagram for Various Velocities

6 Mass and $E = mc^2$

This section has more math than some people may want to read, but I think every every discussion of relativity should at least mention where $E = mc^2$ comes from. Plus, there is one more Lorentz transformation for mass, although it's really a part of the transformation for time. Here's the transformation for a mass m:

$$m(\text{frame with time } \tau) = \gamma m_0(\text{frame with time } t) \tag{7}$$

This is the only transformation where I like the definition of γ, because we remember as a shortcut that "moving masses get heavier". Of course they don't really get heavier; that's just how we measure them from a different reference frame. In fact, this transformation is not even used in newer textbooks. Older books used "relativistic mass" $= \gamma m_0$, where m_0 is the "rest mass". Currently it is more common to leave γ in the equations and drop the subscript "0".

Here's where the mass equations come from. In Newtonian mechanics, we think in terms of position, velocity, and acceleration. For one dimension, this would be x, $\frac{dx}{dt}$, $\frac{dx^2}{dt^2}$. For three dimensions, we go to (x, y, z), etc. The textbooks tell us we should think in terms of *displacement* vectors $(\Delta x, \Delta y, \Delta z)$ instead of position vectors, because a position vector depends on coordinate system, whereas a displacement vector is the same when we change to a different coordinate system in a different reference frame. This is a technical detail that we can leave to the mathematicians for the most part.

In ordinary Euclidean geometry, the distance between two points is calculated from:

$$\text{Distance}^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{8}$$

This distance is the same if you rotate or translate the coordinate system. But as we saw from Alice and Bob's rocket experiments, they get different numbers for the distance between two points and the time passed between two events when they measure them in different reference frames moving with respect to each other.

In relativity, the *interval* between two events is the same in all inertial reference frames, where the interval is defined as:

$$\text{Interval}^2 = S^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{9}$$

Usually we learn relativity starting from the assertion that the speed of light in vacuum is the same in every frame. It is completely equivalent to start with the assertion that the interval is the same in every inertial reference frame.

The negative of the squared interval is the *proper time*:

$$c^2\Delta\tau^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \tag{10}$$

As I hinted above, the proper time is the time measured by a clock in the reference frame of interest. In the rocket experiment, Bob's proper time is what his clock measures. In Bob's frame, he is not moving, so proper time is the only time for him. When Alice observes Bob's clock, and compares it with her own clock, she sees Bob's $\Delta\tau$ is less than her Δt, because she reads Bob's clock at two different places, so the Δx^2 term has to be subtracted.

In relativity, the ordinary three-dimensional vectors of Newtonian mechanics are upgraded to 4-vectors, with a time component. So the displacement vector is ($c\Delta t, \Delta x, \Delta y, \Delta z$).

Then the velocity components become the derivatives of the displacement vector with respect to proper time τ:

$$\text{V} = \left(\frac{cdt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \tag{11}$$

This is the same as ordinary Newtonian physics for the spatial components, as long as velocities are much smaller than the speed of light, because then τ is the same at t, at least to a very good approximation. For the time component, we see that if we're not moving anywhere in space, then we are still moving forward in time at the speed of light. I think that sounds weird at first, but then, everything has to move forward in time. Light sets the pace.

To see what $\frac{dt}{d\tau}$ is, we look at the proper time equation for two events infinitesimally separated. In a frame moving in the x-direction with velocity v, we have:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 \tag{12}$$

$$\frac{c^2 d\tau^2}{c^2 dt^2} = 1 - \frac{1}{c^2}\frac{dx^2}{dt^2} = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} \tag{13}$$

Thus

$$\frac{d\tau}{dt} = \frac{1}{\gamma}, \text{and} \frac{dt}{d\tau} = \gamma \tag{14}$$

From the chain rule, we have:

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma v \tag{15}$$

And for frames moving in the y or z direction, the derivation is the same procedure. The overall result is:

$$\mathrm{V} = \gamma(c, v_x, v_y, v_z) \tag{16}$$

Also we upgrade the definition of the product of two vectors, such as the dot product of a vector with itself, which gives us its magnitude. Everything has to match the definition of the interval. Mathematicians use a metric tensor, but I'll just show the results. The magnitude of the velocity vector is:

$$\mathrm{V} \cdot \mathrm{V} = \left| -c^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 \right| \tag{17}$$

For an inertial rest frame, velocity = 0, $\gamma = 1$, and $dt = d\tau$. So $\mathrm{V} \cdot \mathrm{V} = c^2$. All 4-vectors are invariant, just as the interval is, so $\mathrm{V} \cdot \mathrm{V} = c^2$ in every inertial frame. We can check this for a frame with velocity v in the x direction:

$$\mathrm{V} \cdot \mathrm{V} = \gamma^2(c^2 - v^2) = \frac{1}{1 - \frac{v^2}{c^2}}(c^2 - v^2) = \frac{c^2}{c^2 - v^2}(c^2 - v^2) = c^2 \tag{18}$$

Now if we multiply the velocity 4-vector by mass m, we get the momentum 4-vector, also called the energy-momentum vector or momenergy vector.

$$m\mathrm{V} = \mathrm{P} = \gamma m(c, v_x, v_y, v_z) = m\left(\frac{cdt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) \tag{19}$$

Then we have the $\mathrm{P} \cdot \mathrm{P} = m^2c^2$ in every frame. The space part is ordinary Newtonian momentum multiplied by γ, so that gives us the Lorentz transformation for mass. The final step is to identify the time component, $\mathrm{P}_0 = \gamma mc = m\frac{dt}{d\tau}$, with E/c, that is, energy divided by the speed of light. Why is that? Well, most research involves a lot of fiddling around to see what works, and at the end we write it up as if we knew where to go step by step. We need *something* to be the time component of the 4-vector, and energy divided by velocity has the right units. Next we see that if $E = m\gamma$, then at low velocity:

$$E = mc^2(1 - \frac{v^2}{c^2})^{-1/2} = mc^2 + \frac{1}{2}mv^2 + O(v^4) \tag{20}$$

This gives us the Newtonian kinetic energy with the addition of the famous term mc^2, which is the energy of the mass at rest. When $v = 0$, then $\gamma = 1$ and $\mathrm{P} \cdot \mathrm{P} = m^2c^2 = (\frac{E}{c})^2 = \frac{(mc^2)^2}{c^2} = m^2c^2$. And in a frame moving with velocity v in the x direction,

$$P \cdot P = |\gamma^2(-m^2c^2 + m^2v^2)| = \left|\frac{-m^2(c^2 - v^2)}{1 - \frac{v^2}{c^2}}\right| = \left|\frac{-m^2c^2(c^2 - v^2)}{c^2 - v^2}\right| = m^2c^2 \tag{21}$$

As before, the derivation for a frame moving in the y or z direction is the same procedure.

Now we just substitute $v = 0$ in Equation (20) and get $E = mc^2$ Ta da!

7 Summary

Special relativity introduces a geometry of spacetime that is different from our ordinary experience. When we observe something moving past us at constant velocity close to the speed of light, it appears that the objects in the frame get shorter in the direction of motion and the clocks slow down. Also the momentum appears to increase in the direction of motion. This is sometimes interpreted as an increase in mass, but it is really part of the time effect. These apparent changes are coordinate issues, not real effects. As long as two observers stay in two different reference frames, where one frame moves at constant velocity with respect to the other one, there can be only one event simultaneous for the two observers.

Relativity is not just for rocket scientists; it is also a key part of the explanation for the magnetic force and the spin of electrons, which is important for chemical and electronic properties. Much more information about special relativity is available in textbooks and popular science books. Several recommended books are given in the references.

References

[1] This quote can be found in many places, and is readily available online.

[2] Philosophy of Physics: Space and Time, Tim Maudlin, 2012.

[3] A Student's Guide to Special Relativity, Norman Gray, Cambridge University Press, 2022.

[4] Spacetime Physics: Introduction to Special Relativity, Second Edition, Edwin F. Taylor and John Archibald Wheeler, 1992.

[5] Special Relativity and Classical Field Theory: The Theoretical Minimum, by Leonard Susskind and Art Friedman, 2017.