Lecture Notes on Einstein Acceleration

Linda Fugate

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1 Introduction

Here's where we transition from special to general relativity. Sections 2 and 3 are based mostly on pictures and descriptions, while Sections 4 - 7 include more math.

2 Where Things Get Real

We saw in the twin paradox that people appear to age more slowly when they travel at constant velocity near the speed of light relative to someone else, but this is not real; it's just a coordinate issue. When Bob flies overhead at 60% of the speed of light, he sees me moving at the same speed in the other direction, but nothing changes for me.

Things get real when we add acceleration, so that at least one person can turn around and come back to meet the other one for a second time. Here I want to point out there is no absolute velocity; only velocity relative to something else. Often we take shortcuts and say things like, "When something moves at close to the speed of light, it has relativistic time dilation and length contraction." Something can move at close to the speed of light *only* with respect to something else, and it shows relativistic effects *only* when measured from the reference frame of that something else.

Now, acceleration *is* absolute. Since we have put a lot of energy into learning that velocity has to be relative to something, we can very reasonably wonder, what is acceleration relative to? One answer could be, relative to the total mass of the Universe. Another answer is, relative to any inertial reference frame (and therefore, to all of them).

Isaac Newton realized acceleration must be absolute, and from this he concluded that space and time must be absolute. Einstein said Newton was right about acceleration, but this does not require absolute space and time. Here's one thought experiment. Suppose we have two globes (solid balls) connected with a cord, somewhere in interstellar space. See Figure 1. If they are spinning, the centrifugal force will produce tension in the cord proportional to the rate of spinning. Spin is one form of acceleration, and we don't need any reference points to see whether the globes are spinning.

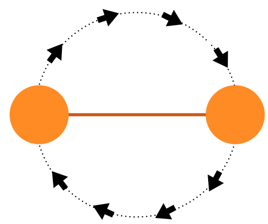

Figure 1: Spinning Globes

Newton also considered a spinning bucket of water in a gravitational field. The surface of the water forms a parabolic shape, with depth proportional to the rate of spinning. The water is not spinning with respect to the bucket, but there is a big difference between spinning water and non-spinning water.

It is well known that fighter jet pilots experience g-forces when they accelerate. See Figure 2. The same effect is used for thrill rides such as roller coasters. The term "g-force" refers to the acceleration of gravity: 1g = 9.8 m/s^2. When a pilot experiences 5 g's, that is equivalent to the force he would feel in a gravitational field 5 times as strong as Earth's.

Figure 2: Fighter Jet

Einstein's general relativity starts with the equivalence principle: acceleration is equivalent to a uniform gravitational field. If Bob and Alice are in an elevator, they can't tell (without high precision measurements) whether they are at rest on the ground floor or out in space with rockets accelerating them at 1g. See Figure 3.

The high precision measurements that could distinguish these two cases are based on

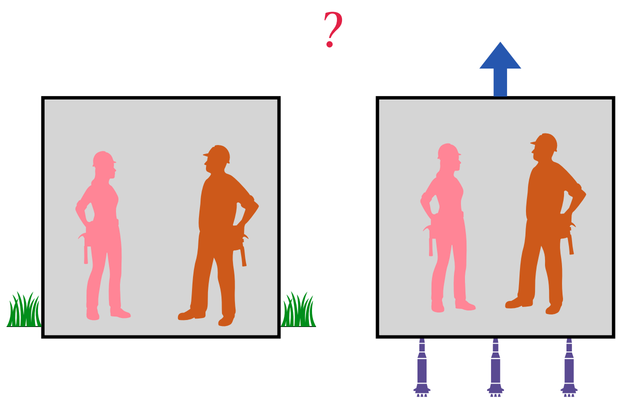

Figure 3: Elevator: at Rest on Earth or Accelerated in Outer Space?

tidal forces, which are caused by gravitational fields that are not perfectly uniform. The equatorial radius of Earth is 6,378.1 km, or 6,368,100 m. So the force of gravity at the surface is:

$$F_S = \frac{GMm}{(6,368,100\text{m})^2} \tag{1}$$

And the force of gravity 2 meters higher is:

$$F_h = \frac{GMm}{(6,368,102\text{m})^2} = \frac{GMm}{(6,368,100\text{m})^2(1+\frac{2}{6368100})^2} \tag{2}$$

From the binomial theorem, $(1+x)^2 \approx 1+2x$ for $x \ll 1$ and $\frac{1}{1+x} \approx 1-x$ for $x \ll 1$. Thus we calculate:

$$F_h = F_S\left(1 - \frac{4}{6363100}\right) = 0.9999994F_S \tag{3}$$

These two numbers are almost identical. In addition, if Bob and Alice both drop an apple in the gravitational field of Earth, both apples will fall toward the center of Earth, so their falling trajectories will be not quite parallel. For uniform acceleration, the apples will fall on parallel paths.

3 Hyperbolic Curves

The world line of an object with constant acceleration on a Minkowski diagram is a curved line, a hyperbola with an asymptote at 45° on a light ray. See Figure 4. Since gravity is equivalent to acceleration, we think of spacetime as curved by gravity.

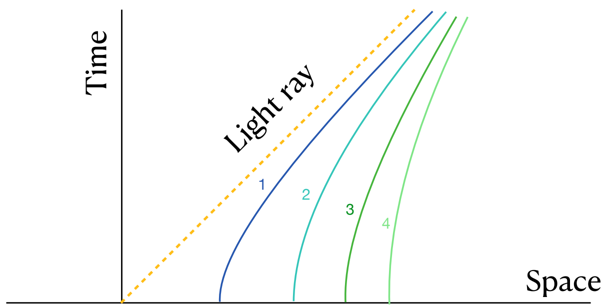

Figure 4: World Lines of Accelerated Objects, from Highest Acceleration (1) to Lowest (4)

There is a "slice" of 4-dimensional spacetime containing events simultaneous with whatever I'm doing in my rest frame. This is sometimes called a "plane of simultaneity", although it's really 3-dimensional space, therefore a hyperplane. Observers in different reference frames, traveling at constant velocity with respect to my reference frame, have different planes of simultaneity. Let's take another look at the Minkiowski diagram for various velocities from special relativity. The world line of the light ray is shown as a dotted yellow line at 45°. The time axes (above the light line) are the world lines of an object or observer moving at the indicated speed relative to an observer in the lab frame with perpendicular black time and space axes. The corresponding space axes (below the light line) are the planes of simultaneity for these moving observers.

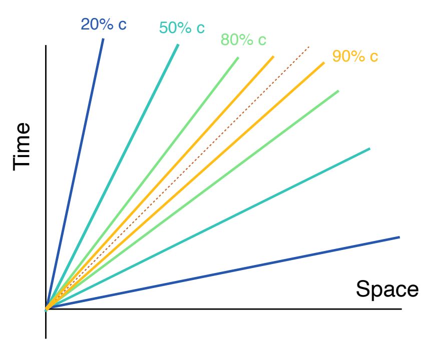

Figure 5: Time and Space Axes in Different Frames, Moving at 20%, 50%, 80%, and 90% of the Speed of Light Relative to the Black Axes

A constantly accelerated observer will have a constantly changing time axis as a tangent to her hyperbolic world line. Therefore, her plane of simultaneity will constantly move closer to the light line in Figure 5. So she must observe clocks in the lab frame ahead of her to run faster. Equivalently, acceleration slows down clocks, and we will see that gravity does the same.

4 Bell's Spaceship Paradox

John Stewart Bell presented a relativity challenge to his colleagues at CERN [1]. Suppose 3 spacecrafts, A, B, and C, are initially at rest with respect to each other. B and C are equidistant from A, and arranged as shown in Figure 6. A thin thread connects B and C.

At time zero, A sends a signal to B and C, and they fire their rockets for a gentle acceleration. They follow exactly the same acceleration program. Will the string break?

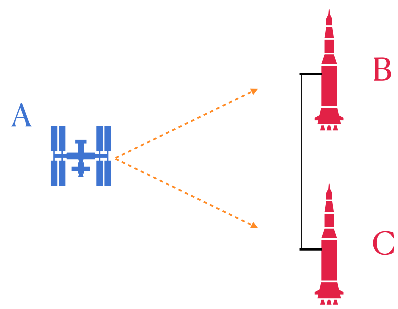

Figure 6: Bell's Spaceship Paradox

Yes, Bell told us. This seems impossible, of course. Bell was not the first to pose this problem, but he was the one to popularize it [2]. In the last section, we saw that acceleration changes the plane of simultaneity, causing accelerated clocks to run slow. This effect is real, just as the different aging rates in the twin paradox. The spaceship paradox is also an effect of the relativity of simultaneity.

Author Leonard Susskind tells us "uniform acceleration" in relativity is different from what we expect from Newtonian mechanics [5]. Consider a series of points equally spaced along the x axis. Now accelerate them all in the x-direction. I think of a long rocket with dots marked along its side. For the points to stay the same distance apart, Susskind tells us the tail of the rocket has to accelerate faster than its nose, as seen from a non-accelerating lab frame. The acceleration of points between the tail and the nose is a smooth function of distance.

Let X and T be lab coordinates. Define accelerated coordinates r and τ by:

$$X = r \cosh \frac{c\tau}{r} \tag{4}$$

$$T = \frac{r}{c} \sinh \frac{c\tau}{r} \tag{5}$$

So

$$X^2 - c^2T^2 = r^2 \left(\cosh^2 \frac{c\tau}{r} - \sinh^2 \frac{c\tau}{r} \right) = r^2 \tag{6}$$

Relativistic acceleration is the 4-vector

$$a^\mu = \frac{d^2 X^\mu}{d\tau^2} \tag{7}$$

So for one-dimensional constant acceleration,

$$|a|^2 = (a^1)^2 - c^2(a^0)^2 = \text{constant} \tag{8}$$

Taking derivatives, for constant r,

$$(a^1) = \frac{d^2 X}{d\tau^2} = r \left(\frac{c}{r} \right)^2 \cosh \frac{c\tau}{r} = \frac{c^2}{r} \cosh \frac{c\tau}{r} \tag{9}$$

$$(a^0) = \frac{d^2 T}{d\tau^2} = \frac{r}{c} \left(\frac{c}{r} \right)^2 \sinh \frac{c\tau}{r} = \frac{c}{r} \sinh \frac{c\tau}{r} \tag{10}$$

And so

$$|a|^2 = (a^1)^2 - c^2(a^0)^2 = \frac{c^4}{r^2} \left(\cosh^2 \frac{c\tau}{r} - \sinh^2 \frac{c\tau}{r} \right) = \frac{c^4}{r^2} \tag{11}$$

$$|a| = \frac{c^2}{r} \tag{12}$$

Thus, for any positive value of r, we have a hyperbola $X^2 - c^2T^2 = r^2$ where the value of acceleration is constant, c^2/r.

Susskind actually does this using $\omega = \frac{c\tau}{r}$. For constant ω, we have

$$\frac{T}{X} = \frac{(r/c) \sinh \omega}{r \cosh \omega} = \frac{1}{c} \tanh \omega = \text{constant} \tag{13}$$

Figure 7 shows lines of constant ω and hyperbolas of constant r in the lab frame X, T. Since r is constant along each hyperbola, the difference in r from one hyperbola to the next is the same along each line of constant ω.

Suppose r_1 is the position of the tail of the rocket, r_4 is the position of its nose, and the other r's are in between them. If we start at the $T = 0$ line (the X axis) and follow the hyperbolas upward, we see how the r's accelerate at constant rates, moving forward (upward) in time and to the right in the X space direction.

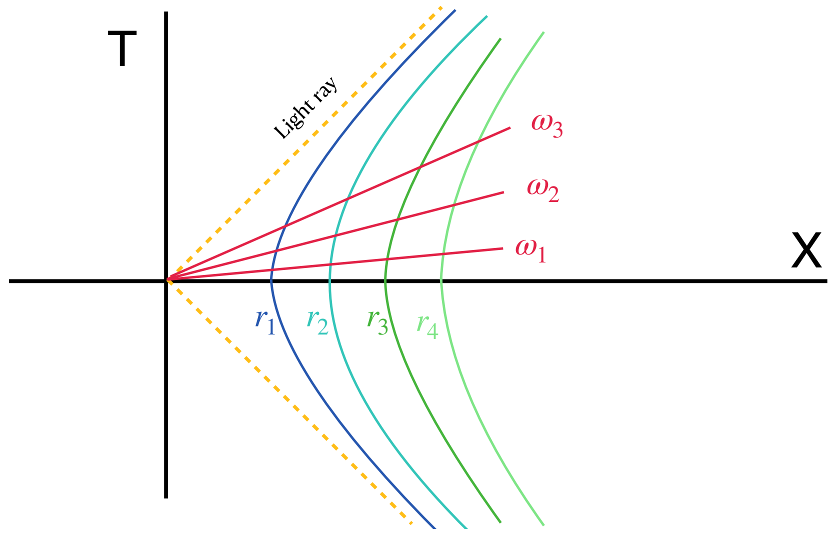

Figure 7: Coordinate Transformation. X and T are space and time in the lab frame. The hyperbolas are curves of constant r. The red lines are lines of constant ω.

Now we recall that the value of acceleration is different on each hyperbola; it is equal to c^2/r. This tells us the tail of the rocket is accelerating faster than its nose. So if we connect a thread from nose to tail, and cut the rocket in half to make two rockets, and speed up the nose to match the tail in the lab frame, then indeed the thread must break.

Figure 8 shows a Minkowski diagram of two accelerating rockets, as in Bell's spaceship paradox, along with an observer stationary in the lab frame. The lead rocket world line hyperbola is shown in red, along with its planes of simultaneity (red straight lines) at four points along the trajectory. As we look ahead (to the right) of the rocket, we see the simultaneous events are spread out, so that the lab frame observer's clock runs faster as seen from the lead rocket.

Now when we look behind (to the left) of the lead rocket, we see the planes of simultaneity converge, so that the lead rocket sees the following rocket lagging behind, with its clock running slower. In the lead rocket's frame, the string breaks because the following rocket is accelerating more slowly, so the distance between them increases.

5 Overview of Curved Space

No doubt everyone has seen pictures of the rubber sheet deformed by a heavy ball. This is only two-dimensional, but it's a pretty good illustration of gravity as a curvature of spacetime. Einstein wrote his field equations by finding a mathematical way to set gravity

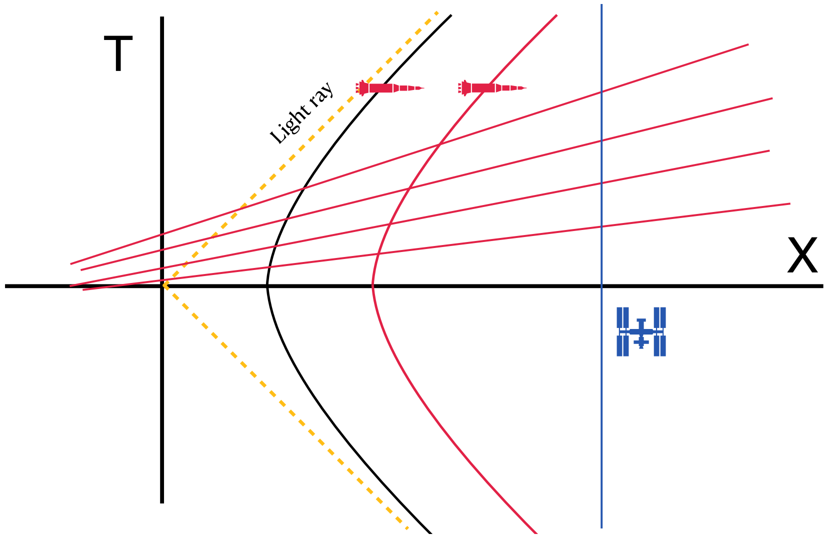

Figure 8: World Lines of Two Accelerating Rockets Plus Observer in Lab Frame. The straight red lines are planes of simultaneity for the rocket in the lead (to the right).

equal to curvature. In special relativity, he used a 4-vector for energy and momentum, and concluded mass is equivalent to energy as in $E = mc^2$. In general relativity, the 4-vector gets upgraded to a 4×4 energy-momentum tensor.

To calculate curvature, Einstein used math methods developed by Bernhard Riemann. All engineers are familiar with spherical coordinates, which are useful for a 2-dimensional surface curved into the third dimension. Gravity can curve all four dimensions of spacetime into an unknown number of higher dimensions. For complicated situations such as the merger of two black holes, this is a highly advanced mathematical problem. Most of the time, however, we consider gravity effects from a spherically symmetric body, such as earth, the sun, or a black hole. In the simplest cases, only two dimensions are curved: time, plus the radial dimension of space. The equations are still not easy to solve, but they are accessible.

The Riemann curvature tensor is a quantitative measure of curvature. It has four indices, so it can't be proportional to the energy-momentum tensor with two indices. According to author Leonard Susskind, Einstein figured out how to make his equations work mostly by fiddling around with the math. David Hilbert discussed relativity with Einstein and then derived the equations from a Lagrangian approach in parallel with Einstein's work, but he gave Einstein credit.

The Riemann tensor can be contracted to form the Ricci tensor, with two indices, and this can be further contracted to form the Ricci scalar. The metric tensor is the solution to the field equations. Einstein put these together as follows:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu} \tag{14}$$

where $R^{\mu\nu}$ is the Ricci tensor, $g^{\mu\nu}$ is the metric tensor, $T^{\mu\nu}$ is the energy momentum tensor, and the rest are scalars. This is a set of differential equations, and the solution is the metric tensor.

Here's what we do with the metric tensor. You should remember that the squared interval between two events in special relativity is given by:

$$dS^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{15}$$

It is different from ordinary Euclidean geometry because it includes time, and the time dimension has a minus sign. For general relativity, first we change to spherical coordinates:

$$dS^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{16}$$

The coefficients of the time and space dimensions form the metric tensor, $g^{\mu\nu}$:

$$g_{mn} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{17}$$

Next we add gravity from a spherically symmetric source, and assume the metric tensor will have the form:

$$g_{mn} = \begin{pmatrix} Something & 0 & 0 & 0 \\ 0 & Something\, else & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{18}$$

Off-diagonal elements will become non-zero in more complicated cases, such as rotating stars or black holes.

6 The 99 Bottles of Beer on the Wall

For many years I heard general relativity is *so* complicated, I thought the math must be exotic. As it turns out, it's just a set of differential equations with an enormous number of terms. Solving Einstein's field equations is like singing "99 Bottles of Beer on the Wall": the same thing over and over with small variations.

The references [3, 4, 5] spell out the field equations and the simplest solution (for the Schwarzschild metric) in great detail. Here I'll just summarize. From the metric tensor, the next step is the Christoffel coefficients Γ, also called connection coefficients:

$$\Gamma^t_{mn} = \frac{1}{2}\sum_{r=0}^{3} g^{rt}\left[\frac{\partial g_{rm}}{\partial x_n} + \frac{\partial g_{rn}}{\partial x_m} - \frac{\partial g_{mn}}{\partial x_r}\right] \tag{19}$$

These are also used to calculate derivatives such as the divergence, curl, gradient, and Laplacian in spherical coordinates [6], although many textbooks use a more geometrical approach and do not mention Christoffel symbols.

From the Christoffel coefficients, we calculate the Riemann curvature tensor:

$$R^t_{srn} = \frac{\partial \Gamma^t_{sn}}{\partial X_r} - \frac{\partial \Gamma^t_{rn}}{\partial X_s} + \sum_{p=0}^{3} \Gamma^p_{sn} \Gamma^t_{pr} - \sum_{p=0}^{3} \Gamma^p_{rn} \Gamma^t_{ps} \tag{20}$$

And the contraction to form the Ricci tensor is:

$$R_{\lambda\beta} = \sum_{\phi=0}^{3} \sum_{\alpha=0}^{3} g^{\alpha\phi} R_{\phi\lambda\alpha\beta} = \sum_{\alpha=0}^{3} R^{\alpha}_{\lambda\alpha\beta} \tag{21}$$

Then the contraction to form the Ricci scalar is:

$$R = \sum_{\alpha} \sum_{\beta} g^{\alpha\beta} R_{\alpha\beta} \tag{22}$$

That's all the terms for the left hand side of Equation (4). The right hand side is the energy momentum tensor, which has all zeros for empty space. The first and easiest solution to the set of equations is the Schwarzschild metric, which assumes a point mass. The solution is valid anywhere outside that one point, where there is assumed to be no mass, energy, or momentum. Thus, the value of the mass comes in only as a boundary condition.

7 Solving Einstein's Equations

Since we assume spherical symmetry, we are looking for only two values of the metric tensor, g_{00} and g_{11}. I would expect this to be not so complicated, but it is. Most of the terms cancel out so that we end up with two fairly simple differential equations, but we have to go through the equivalent of 99 bottles of beer on the wall to find those equations. Maybe someday, some young mathematicians will find a shortcut for us.

Here's an outline of the solution:

1. Assume a solution of the form $g_{00} = -c^2 e^{2\Phi(r)}$, $g_{11} = e^{2\Lambda(r)}$, $g_{22} = r^2$, $g_{33} = r^2 \sin^2 \theta$. The Λ here is not related to the cosmological constant, and the Φ is not related to the angular coordinate; both are just unknown functions of r.

2. Calculate the Christoffel coefficients.

3. Calculate the Riemann curvature tensor, then contract to find the Ricci tensor and contract again to find the Ricci scalar.

4. Substitute into the field equations to get differential equations for the 00 and 11 components.

5. Solve these equations.

6. Compare with Newtonian gravity to get boundary conditions.

Here are the two differential equations:

$$e^{-2\Lambda}\left(\frac{2}{r}\frac{d\Phi}{dr}+\frac{1}{r^2}\right)-\frac{1}{r^2}=0 \tag{23}$$

$$e^{-2\Lambda}\left(\frac{2}{r}\frac{d\Lambda}{dr}-\frac{1}{r^2}\right)+\frac{1}{r^2}=0 \tag{24}$$

The solutions are:

$$g_{00}=-c^2e^{2B}\left(1+\frac{R_0}{r}\right),\; g_{11}=\frac{1}{1+\frac{R_0}{r}} \tag{25}$$

where B and R_0 are integration constants. The boundary condition at large r sets $B=0$, and correspondence with Newtonian gravity for a freely falling body on a radial trajectory sets

$$R_0=\frac{2GM}{c^2} \tag{26}$$

where G is the gravitational constant, M is the mass of the point mass, and c is the speed of light. R_0 is called the Schwarzschild radius, and we will look at its significance in the next lecture.

References

[1] Philosophy of Physics: Space and Time, Tim Maudlin, 2012.

[2] John S. Bell on The Foundations of Quantum Mechanics, M. Bell, K. Gottfried, and M. Veltman, editors, 2001.

[3] A General Relativity Coursebook, Ed Daw, 2023. Written as a text for an undergraduate course.

[4] Einstein's General Theory of Relativity: A Concise Introduction, Brian P. Dolan, 2023.

[5] General Relativity: The Theoretical Minimum, by Leonard Susskind and Andre Cabannes, 2023.

[6] Mathematical Methods for Physicists, George Arfken, 1985.