Twin Paradox: Feynman or Maudlin?

Linda Fugate

February 17, 2024

1 Introduction

The twin paradox is a popular story about Einstein's theory of relativity. Everyone in the physics field agrees on the answer, but there remains some controversy as to why it works that way.

Here's the story: a set of twins live on a space station. Twin A takes a long rocket trip and then returns. Twin B remains home on the space station. Let's call these two Alice and Bob. In relativity theory, moving clocks appear to run slower, so Bob sees Alice aging more slowly. Except for light itself, there is no absolute velocity in relativity theory, only velocity relative to something else, so from Alice's perspective, Bob is the one who traveled away and then returned. When relativity theory first entered popular culture, we started hearing, "Everything is relative."

So when they get together again, is either one really younger than the other?

Yes, according to physics theory. Richard Feynman gave the classic explanation in his text The Feynman Lectures on Physics [Feynman et al., 1963]. Here's what he wrote:

> "So the way to state the rule is to say that the [twin] who has felt the accelerations . . . is the one who would be the younger; that is the difference between them in an "absolute" sense and it is certainly correct."

Acceleration is any change in velocity: any time you speed up, slow down, or change directions, you are accelerating in the physics sense. Acceleration is not relative; it is absolute, and it's an absolutely necessary part of the twin paradox story. As long as Alice moves at constant speed relative to Bob, each will see the other's clocks running slowly, because they have different reference frames. But that's just a coordinate issue: special relativity mixes the time and space coordinates. Time doesn't really slow down for either one. But if they want to get back together to compare how much time has actually passes for each one, then Alice has to accelerate in order to turn around and come back, and when we add acceleration, things get real.

Feynman's quote was published in 1963, and I first read it when I had only a vague knowledge of general relativity. I knew Einstein showed us that acceleration is equivalent to a gravitational field. (Think about the g-forces on fighter pilots.) Clocks run slower in a gravitational field, and the stronger the field, the slower the clock. Near the event horizon of a black hole, time appears to freeze. So I thought sure, clocks run slower in gravitational fields, acceleration is equivalent to gravity, and Alice got accelerated. So that's what made her younger, right?

No, according to other physicists. The person who explained it most clearly, in my opinion, is Tim Maudlin, Professor of Philosophy at NYU [Maudlin, 2015]. He wrote:

> "The Twins "Paradox" has inspired more confusion about Relativity than any other effect. The explanation of the phenomenon, in terms of the intrinsic geometry of Minkowski spacetime and the Clock Hypothesis is exquisitely simple: clocks measure the Interval along their world-lines, and B's world-line between o and q is longer than A's. . . . The accelerations play no role in explaining the end result. Indeed, it is a simple matter to alter the situation so that B is accelerated exactly as much, or even more than A, but still ends up older than A."

What? I was astounded when I read this in Maudlin's book Philosophy of Physics: Space and Time. Reading further, I discovered what he meant.

2 Analyzing the Classic Trip

To make the calculations simple, Maudlin used these details. Alice fires her rockets briefly and accelerates up to 80% of the speed of light. Then she turns off the rockets and coasts at constant speed for 5 years (as measured by Bob). This takes her $80\% \times 5 = 4$ lightyears away from Bob. Then she fires his rockets again and accelerates to 80% of the speed of light in the opposite direction, headed back home. Then she coasts along at constant speed until she arrives home, and fires her rockets one more time to slow down and rejoin Bob. We assume the time the rockets are firing is negligible compared to the total length of the trip.

So from Bob's reference frame, Alice traveled 5 years out and 5 years back, for a total of 10 years, and Bob is 10 years older. Now to calculate how much Alice has aged, we need a little more math. There is an interval formula of special relativity, and the clock hypothesis tells us a clock measures the interval along its world line, which is its path through space and time. Clocks, in this sense, include everything that changes with time, including the biological processes of aging. It comes out to be only 6 years for Alice.

First, let's look at a picture. In relativity, we use Minkowski diagrams where the time axis is vertical and the space axis horizontal. This is different from ordinary graphs; for something like stock prices versus time, we put time on the horizontal axis as the independent variable. In relativity, we want to show how something moves through time as well as space. And we use units where the speed of light is 1; for the twin paradox, we'll

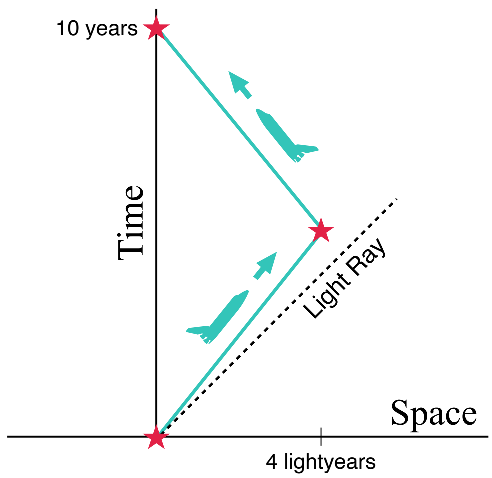

Figure 1: Minkowski Diagram for Alice's Trip

use years for time and lightyears for space distance. So the speed of light is 1 lightyear/year. Figure 1 shows Alice's world line for the trip. She travels 4 lightyears in distance to the right, then 4 lightyears distance to the left, while always moving forward (vertically up) in time.

Bob's world is simple. He doesn't move through space, so he just moves forward (vertically up) in time. His world line is the same as the time axis.

Here is the interval formula for a trip along a straight line:

$$\Delta S^2 = -c^2 \Delta t^2 + \Delta x^2 \tag{1}$$

where S = Spacetime interval, c = speed of light, t = time, x = distance

If we change the signs of space and time, we get the equation for proper time τ, which is the time measured by a clock which makes the trip:

$$c^2 \Delta \tau^2 = c^2 \Delta t^2 - \Delta x^2 = -\Delta S^2 \tag{2}$$

That minus sign in front of the space term in proper time, or time term in the interval, is what gives us all the surprising results of relativity theory. The interval is the same in all reference frames moving at constant velocity (no acceleration); it is invariant. The invariant interval is equivalent to the statement that the speed of light is constant in all reference frames.

Now let's select three events, marked by red stars in Figure 1: Event 1, when Alice blasts off from the space station, Event 2, when she reaches her destination and turns around, and Event 3 when she returns to the space station, where Bob is waiting. The interval between Event 1 and Event 2 is easy to calculate in Bob's reference frame, which is the one pictured in Figure 1, where we see $\Delta x = 4$ lightyears. $\Delta t = 5$ years, since that is the time Bob calculated for Alice to travel 4 lightyears at 80% of the speed of light. Thus we have

$$c^2 \Delta\tau^2 = \left(\frac{1 \text{ lightyear}}{\text{year}} \times 5 \text{ years} \right)^2 - (4 \text{ lightyears})^2 \tag{3}$$

This gives us $\Delta\tau = 3$ years. That is the time measured by Alice's clocks. The calculation for the return trip between Event 2 and Event 3 is the same, so on Alice's clocks, the trip takes 3 years each way, for a total of 6 years. There is also a relativistic length contraction which means that Alice does not see her destination moving toward her at more than the speed of light. For the twin paradox story, all we need to see is that Alice ages 6 years while Bob ages 10 years.

We did not explicitly include acceleration in our calculations. Maudlin wrote that each twin could experience the same acceleration during Alice's trip. Suppose Bob makes a short trip out to the galactic grocery store: he could accelerate up to 80% of the speed of light, coast for a few minutes, decelerate, pick up his groceries curbside, accelerate to 80% c for the return trip, and decelerate to dock at the space station again. Then his acceleration history would be the same as Alice's, but his world line (path through spacetime) would not change much, and he would still be essentially 4 years older than Alice.

3 Analyzing a Modern Trip

Acceleration for travel over a distance of a few lightyears is in a different league from any technology we have available. If Alice could accelerate from 0 to 80% of the speed of light in a time much shorter than her trip, say 10,000 seconds, the g-forces would be a few thousand times what fighter pilots experience. That would not only kill the astronaut; it would crush the rocket as well.

Science fiction writers such as Andy Weir [Weir, 2021] offer a different scenario. For a modern trip, let's put Bob back on Earth with its 1g gravitational field. This time Alice accelerates at 1g until she reaches the half-way point to her destination at 4 lightyears distance. Then she cuts the engines and turns the rocket around. Next she restarts the engines and decelerates at 1g until she arrives at her destination. After a brief stop, she repeats this procedure to return home to Earth. This trip requires an enormous amount of fuel, but then, the classic trip would also require phenomenal energy to produce the near-instantaneous acceleration.

How long does this trip take, according to Bob's clocks on Earth and according to Alice's clocks in the rocket? The formulas are worked out in [Blennow and Ohlsson, 2022];

I will just quote them here. In Bob's frame, the distance Alice travels as a function of time at constant acceleration g is:

$$x = \frac{c^2}{g}\left[\sqrt{1+\left(\frac{gt}{c}\right)^2} - 1\right] \tag{4}$$

Here we want to use the distance of 2 lightyears, since Alice accelerates constantly to the halfway point. Solving for t, we get:

$$t = \frac{c}{g}\sqrt{\left(\frac{xg}{c^2}+1\right)^2 - 1} \tag{5}$$

Blennow and Ohlsson give us the very helpful unit conversion $g = 9.81$ meters/second^2 = 1.05 lightyears/year^2. So with $c = 1$ lightyear/year, we have

$$t = \frac{1}{1.05} \times \sqrt{\left(\frac{2 \times 1.05}{1^2}+1\right)^2 - 1} = 2.79 \text{ year} \tag{6}$$

The total trip has 4 segments of constant acceleration, so the time elapsed for Bob is $4 \times 2.79 = 11.2$ years. For Alice's proper time, Blennow and Ohlsson give the formula:

$$\tau = \left(\frac{c}{g}\right) \operatorname{arccosh}\left(1 + \frac{gx}{c^2}\right) \tag{7}$$

The inverse hyperbolic cosine function can be expanded as $\operatorname{arccosh}(\mathrm{u}) = \ln\left(u + \sqrt{u^2-1}\right)$ for $u \geq 1$. To get Alice's time for one segment of the trip, we set $u = 1 + (1.05 \times 2)/1^2 = 3.1$, and calculate:

$$\tau = \left(\frac{1}{1.05}\right) \ln\left(3.1 + \sqrt{3.1^2 - 1}\right) = 1.71 \tag{8}$$

The total trip has 4 segments of constant acceleration, so Alice's time is $4 \times 1.71 = 6.84$ years. The age difference caused by this trip is in the ballpark with the age difference of the classic trip.

Reference Frame	Time for Classic Trip	Time for Modern Trip
Bob (stay)	10 years	11.2 years
Alice (travel)	6 years	6.8 years

Figure 2 shows a Minkowski diagram comparing the two trips.

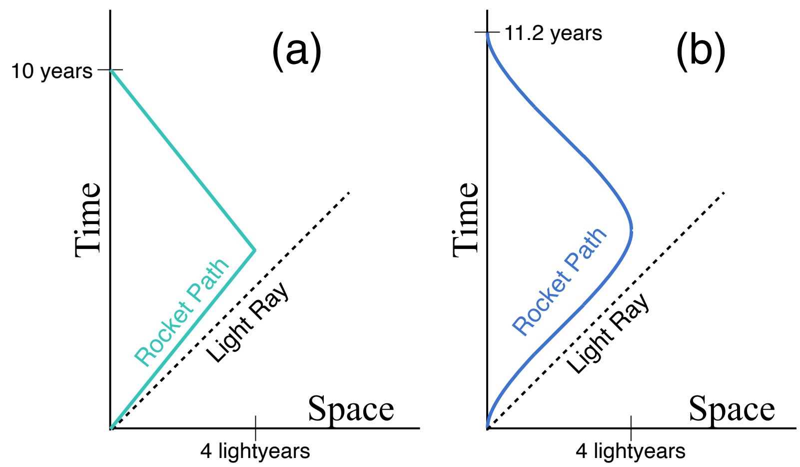

Figure 2: Minkowski Diagram Comparing Two Possibilities for Alice's Trip. (a) Classic trip with instantaneous acceleration to 80% of the speed of light. (b) Modern trip with acceleration and deceleration of 1g.

4 Effect of Gravity

In relativity, a perfect inertial reference frame has no acceleration and no gravity. Since the force of gravity never goes to zero, there is no absolutely perfect inertial reference frame. However, a rocket coasting in interstellar space is close to such a perfect frame. Earth is also remarkably close. We can calculate the relativistic effect of Earth's 1g gravity by using the Schwarzschild metric for the interval between two events:

$$\Delta S^2 = -\left(1 - \frac{R_S}{r}\right) c^2 \Delta t^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \Delta r^2 + r^2 \Delta\Omega^2 \tag{9}$$

where R_S is the Schwarzschild radius, and $\Delta\Omega$ is an angular term. For two events both happening on Earth's surface, Δr and $\Delta\Omega$ are both zero.

The radius of Earth is 6,378 kilometers, and its Schwarzschild radius is 8.87×10^{-3} m. So the proper time between two events on Earth's surface is calculated from:

$$c^2 \Delta\tau^2 = -\Delta S^2 = c^2 \times \left(1 - \frac{8.87 \times 10^{-3}\ \text{meters}}{6.378 \times 10^{3}\ \text{meters}}\right) \times \Delta t \tag{10}$$

Solving for proper time,

$$\Delta\tau = \sqrt{(1 - 1.39 \times 10^{-6})} \times \Delta t = 0.9999998 \Delta t \tag{11}$$

This means the relativistic correction for Earth's gravity is very small. It is necessary to make the correction for the GPS system, because GPS uses signals traveling at the speed of light to measure distance. However, we don't need it for the twin paradox story because it would be too small to measure for 1g. Gravity and acceleration give us the same time dilation effect; both twins in the modern story experience 1g of acceleration throughout the trip, so acceleration can not be the cause of their difference in aging.

Assume hypothetically that Alice and her rocket could survive accelerations large enough to go from zero to 80% of the speed of light almost instantaneously, as in the classic trip. In that case, her clocks would slow down during the time of acceleration, but this would still amount to a tiny change in how much she ages over the entire trip, because the higher the acceleration, the shorter the time it takes to reach cruising speed. Suppose, as we assumed above, it takes 10,000 seconds to reach 80% of c. This is about 3 hours. Then it would also take 3 hours to decelerate at her destination, then 3 hours to accelerate for the trip home, then 3 hours to decelerate and land. If her clocks stopped completely during those 12 hours of acceleration time, she would be only 12 hours younger from this effect. Again, the effect of acceleration is very small compared to the effects of the path through spacetime.

5 Conclusions: Was Feynman Wrong?

After my encounter with Maudlin's book, I had to go back and re-read what Feynman said. He never actually told us acceleration caused the traveling twin to age slower. I'm sure he knew how relativity works; he just simplified the story because he wanted to emphasize the fact that not all motion is relative. Constant acceleration has the same effect as a uniform gravitational field, so it can't be "relative" the way constant velocity is. So I don't think Feynman was wrong, I just think he confused the heck out of a generation of young physicists. These things happen when we take shortcuts in trying to explain physics to the public or to undergraduate students. The twin paradox should be a lesson to all us science writers.

References

Matthas Blennow and Tommy Ohlsson. *300 Problems in Special and General Relativity with Complete Solutions.* Cambridge University Press, 2022.

Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics*, volume 1. Addison-Wesley Publishing, 1963.

Tim Maudlin. *Philosophy of Physics: Space and Time.* Princeton University Press, 2015.

Andy Weir. *Project Hail Mary: A Novel.* Ballantine Books, 2021.